UNIVERSITY OF PENNSYLVANIA

# THE RESOLVENTS OF KÖNIG AND OTHER TYPES OF SYMMETRIC FUNCTIONS

BY

STANLEY P. SHUGERT

A THESIS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

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# THE RESOLVENTS OF KÖNIG AND OTHER TYPES OF SYMMETRIC FUNCTIONS.

By Stanley P. Shugert.

## INTRODUCTION.

Symmetric functions of the differences of the roots of an equation are an important part of many theories, particularly of binary invariant theory, and much attention has been given to the computation of such functions; while the functions which are not symmetric in differences have received very little attention, but in some instances have an important bearing in general theories. As a particular instance, there are the resolvents of degree $\binom{m}{r}$ of an equation,

$$f(x) = x^m + A_1x^{m-1} + A_2x^{m-2} + \cdots + A_m = 0,$$

which were studied by J. König in the *Mathematische Annalen*, Vol. 15 (1879). In brief, these are constructed as follows: the roots of

$$F(z) = f(z + \mu) = z^m + B_1z^{m-1} + \cdots + B_m = 0$$

are $(\alpha_i - \mu)$ $\{i = 1, 2, \cdots m\}$, where the $\alpha_i$ are the roots of $f(x) = 0$. The equation whose roots are the $\binom{m}{r}$ quantities of the type,

$$(-1)^r(\alpha_1 - \mu)(\alpha_2 - \mu) \cdots (\alpha_r - \mu) \quad (r = 1, 2, \cdots m),$$

is rational in $B_i$ and is a resolvent of $f(x) = 0$.

In volume 15 (1914) of the *Transactions of the American Mathematical Society*, Prof. O. E. Glenn has published a paper in which is given the expansion of the binary form of order $m = \mu\nu$,

$$f(x) = a_0x_1^m + a_1x_1^{m-1}x_2 + \cdots + a_mx_2^m$$

as a power series in which the argument is the binary form of order $\nu$,

$$\xi_x^\nu = \xi_0x_1^\nu + \xi_1x_1^{\nu-1}x_2 + \cdots + \xi_\nu x_2^\nu.$$

In § 2 of this paper, we obtain an independent proof of his result

for $\nu = 2$ by a method depending upon the solution of a system of linear equations and a preliminary lemma on determinants, § 1. The remainder of the present paper is devoted to a succession of applications, introducing theories, which depend upon this expansion for the case $\nu = 2$, and finally introducing in § 6 a method for the determination of the König Resolvents with the aid of the tables given in § 7. In § 3 a minimum set of seminvariant conditions that a given quadratic form may divide a given form $f$ is obtained and § 4 is devoted to a problem in rational fractions. There are obtained in § 5, methods for the computation of symmetric functions of the sums and of the products of the roots of an equation, taken in pairs.

## § 1. A LEMMA ON DETERMINANTS.

We first prove the following lemma concerning determinants:

LEMMA: *The determinant* $\{a_{rs}\}$ *in which every constituent is zero for* $s > r + 1$ *and not zero for* $s = r$ *can be reduced to a determinant* $\{a_{rs}\}$ *in which every constituent is zero for* $s > r$.

Let $\{a_{rs}\} =$

$$\text{(A)} \qquad \begin{vmatrix} a_{11}, & a_{12}, & 0 & \cdots & 0, & 0 & \cdots & 0 \\ a_{21}, & a_{22}, & a_{23}, & \cdots & 0, & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & \\ a_{i1}, & a_{i2}, & \cdots & \cdots & a_{i\,i+1}, & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & 0 \\ a_{r-1\,1}, & a_{r-1\,2}, & a_{r-1\,3}, & \cdots & a_{r-1\,i+1}, & & \cdots & a_{r-1\,r} \\ a_{r\,1} & a_{r\,2}, & a_{r\,3}, & \cdots & a_{r\,i+1}, & & \cdots & a_{r\,r} \end{vmatrix}.$$

Multiply the last row by $a_{r-1\,r}/a_{r\,r}$ and subtract from the preceding row. This will give

$$\text{(B)} \qquad \begin{vmatrix} a_{11}, & a_{21} & 0, & \cdots & 0, & 0 & \cdots & 0 \\ a_{21}, & a_{22} & a_{23}, & \cdots & 0, & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ a_{i1}, & a_{i2} & \cdots & \cdots & a_{i\,i+1}, & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \dfrac{(a_{r-1}(a_r)_r)_1}{a_{rr}}, & \dfrac{(a_{r-1}(a_r)_r)_2}{a_{rr}}, & \cdots & \dfrac{(a_{r-1}(a_r)_r)_{i+1}}{a_{rr}}, & \cdots & \dfrac{(a_{r-1}(a_r)_r)_{r-1}}{a_{rr}}, & & 0 \\ a_{r1}, & a_{r2}, & \cdots & a_{r\,i+1}, & \cdots & a_{r\,r-1}, & & a_{rr} \end{vmatrix}$$

in which $(a_{r-1}(a_r)_r)_{i+1}$ is a symbolic notation for the second order

determinant $(a_{r-1\,i+1}a_{rr} - a_{r-1\,r}a_{r\,i+1})$. The minor determinant $A_{rr}$ is of the same type as the determinant $A$ and by applying the same process to it and continuing the method for each succeeding minor determinant, we finally obtain:

$$\begin{vmatrix} \dfrac{(a_1(a_2(a_3 \cdots (a_r)_r \cdots)_3)_2)_1}{a_2(a_3 \cdots (a_r)_r \cdots)_3)_2}, & 0, & \cdots 0 \\ \dfrac{(a_2(a_3(a_4 \cdots (a_r)_r \cdots)_4)_3)_1}{(a_3(a_4 \cdots (a_r)_r \cdots)_4)_3}, & \dfrac{(a_2(a_3 \cdots (a_r)_r \cdots)_3)_2}{(a_3(a_4 \cdots (a_r)_r \cdots)_4)_3}, & \cdots 0 \\ \cdots & \cdots & \cdots \\ \dfrac{a_i(a_{i+1} \cdots (a_r)_r \cdots)_{i+1})_1}{(a_{i+1}(a_{i+2} \cdots (a_r)_r \cdots)_{i+2})_{i+1}}, & \dfrac{(a_i(a_{i+1} \cdots (a_r)_r \cdots)_{i+1})_2}{a_{i+1}(a_{i+2} \cdots (a_r)_r \cdots)_{i+2})_{i+1}}, & \cdots \dfrac{a_i(a_{i+1} \cdots (a_r)_r \cdots)_{i+1})_j}{(a_{i+1}(a_{i+2} \cdots (a_r)_r \cdots)_{i+2})_{i+1}}, \cdots 0 \\ \cdots & \cdots & \cdots \\ \dfrac{(a_{r-1}(a_r)_r)_1}{a_{rr}}, & \dfrac{a_{r-1}(a_r)_r)_2}{a_{rr}}, & \cdots 0 \\ a_{r1}, & a_{r2}, & \cdots a_{rr} \end{vmatrix}$$

$(i = 1, 2, \cdots r,\ j = 1, 2, \cdots i)$.

The interlacing determinant notation is merely an extension of the symbolic method explained above for a determinant of the second order. These determinants are expanded by means of a succession of second order determinants and the method as shown for $(a_{r-1}(a_r)_r)_{i+1} = (a_{r-1\,i+1}a_{rr} - a_{r-ir}a_{ri+1})$ is to take the determinant subscripts, those written outside, first in an order symmetric with respect to the center with the product of the constituents and then to subtract the product obtained by reversing the determinant subscripts. Except in the last reduction, two constituents of the second order determinants will be symbolic determinants.

COROLLARY: *The determinant* $\{a_{rs}\}_{s \geqslant r+1}$ *can be written as a symbolic determinant*: *i. e.*

$$\{a_{rs}\}_{s \geqslant r+1} = (a_1(a_2(a_3(a_4 \cdots (a_r)_r \cdots)_4)_3)_2)_1.$$

## § 2. EXPANSION OF A POLYNOMIAL AS A POWER SERIES OF A QUADRATIC POLYNOMIAL.

THEOREM: *Any form of degree m can be expressed in terms of a power series of a quadratic form with linear expressions for the coefficients.*

First: if $m$ is even, $m = 2n$, we may write,

$$a_x^m = a_0x^{2n} + a_1x^{2n-1} + a_2x^{2n-2} + \cdots + a_{2n} = (\alpha_1 + \alpha_2x)$$

$$(1) \qquad + (\alpha_3 + \alpha_4x)\xi_x^2 + \cdots + (\alpha_{2r-1} + \alpha_{2r}x)(\xi_x^2)^{r-1} +$$

$$\cdots + (\alpha_{2n-1} + \alpha_{2n}x)(\xi_x^2)^{n-1} + (\xi_x^2)^n,$$

where $\xi_x^2 = \xi_0x^2 + \xi_1x + \xi_2$.

Sinee $a_0 = \xi_0^n$ and is determined, the number of $\alpha$'s on the right is equal to the number of $a$'s on the left and since the eoeffieients of the powers of $x$ on the right are linear in the $\alpha$'s, we have upon equating coeffieients $2n = m$ linear non-homogeneous equations in the $2n$ quantities $\alpha_i$.

For the case $2n = 4$, we have

$$(a_0a_1a_2a_3a_4)(x)^4 = (\alpha_1 + \alpha_2x) + (\alpha_3 + \alpha_4x)(\xi_0x^2 + \xi_1x + \xi_2) + (\xi_0x^2 + \xi_1x + \xi_2)^2.$$

and upon multiplying out and rearranging,

$$\begin{aligned} \xi_0\alpha_4 &= a_1 - 2\xi_0\xi_1, \\ \xi_0\alpha_3 + \xi_1\alpha_4 &= a_2 - \xi_1^2 - 2\xi_0\xi_2, \\ \alpha_2 + \xi_1\alpha_3 + \xi_2\alpha_4 &= a_3 - 2\xi_1\xi_2, \\ \alpha_1 + \qquad \xi_2\alpha_3 \qquad &= a_4 - \xi_2^2. \end{aligned}$$

Seeond: If $m$ is odd $m = 2n - 1$, we ean write

$$a_x^m = a_0x^{2n-1} + a_1x^{2n-2} + a_2x^{2n-3} + \cdots + a_{2n-1}$$

$$(1a) \qquad = (\alpha_1 + \alpha_2x) + (\alpha_3 + \alpha_4x)\xi_x^2 + \cdots$$

$$+ (\alpha_{2r-1} + \alpha_{2r}x)(\xi_x^2)^{r-1} + \cdots + (\alpha_{2n-1} + kx)(\xi_x^2)^{n-1}.$$

where $k$ is determined to be $a_0/\xi_0^{n-1}$ and we have upon equating

coefficients $2n-1$ linear non-homogeneous equations to determine the $2n-1$ $\alpha$'s.

For the case $m=3$, we have

$$(a_0a_1a_2a_3)(x)^3 = (\alpha_1+\alpha_2x)+(\alpha_3+kx)(\xi_0x^2+\xi_1x+\xi_2)$$

and upon multiplying out and rearranging,

$$\begin{aligned} \xi_0\alpha_3 &= a_1-\xi_1, \\ \alpha_2+\xi_1\alpha_3 &= a_2-\xi_2, \\ \alpha_1+\quad \xi_2\alpha_3 &= a_3, \end{aligned}$$

if, as before, we take $a_0=\xi_0^{n-1}$, which we may do without loss of generality.

The particular form that these equations take, enable us to express them in general in terms of an operator, when we observe that any power of a form is a covariant of that form and so is given by its end term and a partial derivative operator. Hence the equations:

$$D^0\xi_0^{E[(m-1)/2]}\alpha_m = a_1 - D\xi_0^{E(m/2)},$$

$$D^0\xi_0^{E[(m-2)/2]}\alpha_{m-1} + D\xi_0^{E[(m-1)/2]}\alpha_m = a_2 - D^2\xi_0^{E(m/2)},$$

. . . . . .

$$\cdots + D^{i-j}\xi_0^{E[(m-j)/2]}\alpha_{m-j+1}\cdots$$

$$+ D^{i-2}\xi_0^{E[(m-2)/2]}\alpha_{m-1} + D^{i-1}\xi_0^{E[(m-1)/2]}\alpha_m = a_i - D^i\xi_0^{E(m/2)},$$

. . . . . .

$$D^0\xi_0{}^0\alpha_2 + \cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$D^0\xi_0{}^0\alpha_1 + D\xi_0{}^0\alpha_2\cdots + D^{i-j}\xi_0^{E[(m-j)/2]}\alpha_{m-j+1}\cdots$$

$$+ D^{m-2}\xi_0^{E[(m-2)/2]}\alpha_{m-1} + D^{m-1}\xi_0^{E[(m-1)/2]}\alpha_m = a_m - D^m\xi_0^{E(m/2)}$$

$(i=1, 2, 3, \cdots m)$ and $(j=1, 2, 3, \cdots m)$.

Here $E\left(\frac{m-j}{2}\right)$ is the integral part of $\frac{m-j}{2}$, and $D$ is the Aronhold operator

$$\left(D = \xi_1\frac{\partial}{\partial\xi_0} + \xi_2\frac{\partial}{\partial\xi_1}\right).$$

Also $D^{i-j}\xi_0^{E[(m-1)/2]} = 0$ for $j > i$.

Solving for $\alpha_{m-j+1}$, we have

$$\alpha_{m-j+1} =$$

$$\left|\begin{array}{ccccccc}
0, & 0, & \cdots & \cdots, & a_1 - D\xi_0^{E(m/2)}, \\
0, & 0, & \cdots & \cdots, & a_2 - D^2\xi_0^{E(m/2)}, \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0, & 0, & \cdots D^{i-(j+1)}\xi_0^{E[(m-(j+1))/2]}, & a_i - D^i\xi_0^{E(m/2)}, \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0, & D^{m-m}\xi_0^0, & \cdots & \cdots & \cdots \\
D^{m-m}\xi_0^0, & D^{m-(m-1)}\xi_0, & \cdots D^{m-(j+1)}\xi_0^{E[(m-(j+1))/2]}, & a_m - D^m\xi_0^{E(m/2)},
\end{array}\right.$$

$$\left.\begin{array}{ccc}
\cdots & \cdots & 0, \quad D^0\xi_0^{E[(m-1)/2]} \\
\cdots & \cdots & D^0\xi_0^{E[(m-2)/2]}, \quad D^1\xi_0^{E[(m-1)/2]} \\
\cdots & \cdots & \cdots \\
D^{i-(j-1)}\xi_0^{E[(m-(j-1))/2]}, & \cdots D^{i-2}\xi_0^{E[(m-2)/2]}, & D^{i-1}\xi_0^{E[(m-1)/2]} \\
\cdots & \cdots & \cdots \\
D^{m-(j-1)}\xi_0^{E[(m-(j-1))/2]}, & \cdots D^{m-2}\xi_0^{E[(m-2)/2]}, & D^{m-1}\xi_0^{E[(m-1)/2]}
\end{array}\right| \div \Delta,$$

where $\Delta = \xi_0^m$.

If we expand the numerator of this expression in terms of the eolumn $a_i - D^i\xi_0^{E(m/2)}$, the minors of the eonstituents $a_i - D^i\xi_0^{E(m/2)}$ are determinants of the type $(A)$ of § 1. So in the symbolie notation,

$$(2) \quad \alpha_{m-j+1} =$$

$$\frac{(-1)^{i+j}\sum_{i=1}^{j}\{(D^1(D^2\cdots(D^{i-1}(D^{i+1}\cdots(D^{m-1}(D^m)^m\xi_0^0)^{m-1}\xi_0^0 \cdots)^{j+1}\xi_0^{E[(m-(j+1))/2]})^{j-1}\xi_0^{E[(m-(j-1))/2]} \cdots)^2\xi_0^{E[(m-1)/2]})^1\xi_0^{E[(m-1)/2]}\}(a_i - D^i\xi_0^{E(m/2)})}{\xi_0^m}.$$

This result enables us to make a rapid eomputation of the $\alpha_i$ in partieular eases.

Example 1. $m = 4, j = 3$.

$$\alpha_2 = [\{(D^2(D^3(D^4)^4\xi_0^0)^2\xi_0)^1\xi_0\}(a_1 - D\xi_0^2)$$
$$- \{(D^1(D^3(D^4)^4\xi_0^0)^2\xi_0)^1\xi_0\}(a_2 - D^2\xi_0^2)$$
$$+ \{(D^1(D^2(D^4)^4\xi_0^0)^2\xi_0)^1\xi_0\}(a_3 - D^3\xi_0^2)] \div \xi_0^4.$$
$$= [\{D^0\xi_0^0(D^2(D^3)^2\xi_0)^1\xi_0\}(a_1 - D^1\xi_0^2)$$

$$
\begin{aligned}
&\quad - D^0\xi_0{}^0 D^0\xi_0 (D^3)^2\xi_0 (a_2 - D^2\xi_0{}^2) \\
&\quad + D^0\xi_0{}^0 D^0\xi_0 D^0\xi_0 (a_3 - D^3\xi_0{}^2)]^* \div \xi_0{}^4 \\
&= [1(D\xi_0 \cdot D\xi_0 - D^0\xi_0 D^2\xi_0)(a_1 - D\xi_0{}^2) \\
&\quad - 1\cdot\xi_0 \cdot D\xi_0 (a_2 - D^2\xi_0{}^2) + 1\cdot\xi_0\xi_0 (a_3 - D^3\xi_0{}^2)] \div \xi_0{}^4 \\
&= [(\xi_1{}^2 - \xi_0\xi_2)(a_1 - 2\xi_0\xi_1) - \xi_0\xi_1 (a_2 - \xi_1{}^2 - \xi_0\xi_2) \\
&\quad + \xi_0{}^2 (a_3 - \xi_1\xi_2)] \div \xi_0{}^4 \\
&= [(-\xi_0\xi_1{}^3 + a_1\xi_1{}^2 - \xi_0\xi_1 a_2 + \xi_0{}^2 a_3) + 2(\xi_0{}^2\xi_1 - \xi_0 a_1)\xi_2] \div \xi_0{}^4.
\end{aligned}
$$

Example 2. $m = 3, j = 3$.

$$
\begin{aligned}
\alpha_1 &= [(D^2(D^3)^2\xi_0{}^0)^1\xi_0 (a_1 - D\xi_0) - (D^1(D^3)^2\xi_0{}^0)^1\xi_0)(a_2 - D^2\xi_0) \\
&\qquad + (D^1(D^2)^2\xi_0{}^0)^1\xi_0 (a_3 - D^3\xi_0)] \div \xi_0{}^3 \\
&= [(D\xi_0 \cdot D\xi_0{}^0 - D^0\xi_0{}^0 D^2\xi_0)(a_1 - D\xi_0) \\
&\quad - D^0\xi_0 \cdot D\xi_0{}^0 (a_2 - D^2\xi_0) + D^0\xi_0 \cdot D^0\xi_0{}^0 (a_3 - D^3\xi_0)] \div \xi_0{}^3 \\
&= [(\xi_1 \cdot 0 - 1\cdot\xi_2)(a_1 - \xi_1) - \xi_0 \cdot 0(a_2 - \xi_2) \\
&\qquad + \xi_0 \cdot 1(a_3 - 0)] \div \xi_0{}^3 \\
&= [\xi_0 a_3 - (-\xi_1 + a_1)\xi_2] \div \xi_0{}^3.
\end{aligned}
$$

Without interfering with the generality of the results we may take $\xi_0 = 1$, and then the values for $\alpha_1$ and $\alpha_2$ can be put in the following form:†

$$
(3) \qquad
\begin{aligned}
\alpha_1 &= a_m - \sum_{i=0}^{E[(m-1)/2]} \frac{\xi_2{}^{i+1}}{\lfloor i} \Omega_m{}^{i+2} a_{-\xi_1}{}^{m(i)}, \ddagger \\
\alpha_2 &= \sum_{i=0}^{E[(m-1)/2]} \frac{\xi_2{}^{i}}{\lfloor i} \Omega_m{}^{i+1} a_{-\xi_1}{}^{m(i)},
\end{aligned}
$$

where

$$
\Omega_m = a_0 \frac{\partial}{\partial a_1} + a_1 \frac{\partial}{\partial a_2} + \cdots + a_{m-1} \frac{\partial}{a_m}.
$$

Thus, for example, the results in the partieular case, $m = 4$,

* $(D^i)^i\xi_0{}^j = D^{i-i}\xi_0{}^j = D^0\xi_0{}^j$ is factored out of each term since it enters the result as a constant multiplier.

† Cf. *Transactions Amer. Math. Soc.*, Vol. 15 (1914), No. 1, p. 79.

‡ $-\xi_1$ is substituted for $x$ before the differentiation is performed and $(i)$ is the power of the differential coefficient.

can be written

$$\alpha_1 = a_4 - \sum_{i=0}^{1} \frac{\xi_2^{i+1}}{\lfloor i} \Omega_4^{i+2} a_{-\xi_1}^{4^{(i)}},$$

$$\alpha_2 = \sum_{i=0}^{1} \frac{\xi_2^{i}}{\lfloor i} \Omega_4^{i+1} a_{-\xi_1}^{4^{(i)}}, \tag{4}$$

or

$$\begin{aligned} \alpha_1 &= a_4 - \xi_2 \Omega_4^2 a_{-\xi_1}^4 - \xi_2^2 \Omega_4^3 a_{-\xi_1}^{4^{(1)}} \\ &= a_4 - (a_0\xi_1^2 - a_1\xi_1 + a_2)\xi_2 + a_0\xi_2^2, \\ \alpha_2 &= \Omega_4 a_{-\xi_1}^4 + \xi_2 \Omega_4^2 a_{-\xi_1}^{4^{(1)}} \\ &= (-a_0\xi_1^3 + a_1\xi_1^2 - a_2\xi_1 + a_3) + (2a_0\xi_1 - a_1)\xi_2. \end{aligned} \tag{4a}$$

This result agrees with that of example 1 if we make $\xi_0 = 1$. From (3)

$$F = (\alpha_1 + \alpha_2 x) = a_m + \sum_{i=0}^{E[(m-1)/2]} \frac{-\xi_2^{i+1}}{\lfloor i} \Omega_m^{i+2} a_{-\xi_1}^{m^{(i)}} + \frac{\xi_2^{i}}{\lfloor i} \Omega_m^{i+1} a_{-\xi_1}^{m^{(i)}} x. \tag{5}$$

There is proved in the paper just cited a general result* which is applied here for the case $\nu = 2$; namely,

$$(\alpha_{2k+1} + \alpha_{2k+2}x) = -\frac{1}{k}\frac{\partial}{\partial \xi_2}(\alpha_{2k-1} + \alpha_{2k}x).$$

Hence the assumed expansions (1) and (1a) take the form

$$\begin{aligned} a_x^m = F &- \frac{\partial F}{\partial \xi_2}(\xi_x^2) + \frac{1}{\lfloor 2}\frac{\partial^2 F}{\partial \xi_2^2}(\xi_x^2)^2 + \\ &\cdots + \frac{(-1)^k}{\lfloor k}\frac{\partial^k F}{\partial^k \xi_2}(\xi_x^2)^k + \cdots - \frac{(-1)^{E(m/2)}}{\lfloor E(m/2)}\frac{\partial^{E(m/2)}}{\partial \xi_2^{E(m/2)}}(\xi_x^2)^{E(m/2)}. \end{aligned} \tag{6}$$

The coefficients in this expansion are symmetric functions of the roots of $\xi_x^2$ and also of those of $a_x^m$. Expressed in terms of the coefficients of $a_x^m$, and $\xi_x^2$, they are linear in $a_0, a_1, \cdots, a_m$, but, in general, not linear in $\xi_0, \xi_1, \xi_2$. These functions have a seminvariant character as appears in the application which follows.

* In Glenn's paper the corresponding expansions of the general form of order $\mu\nu$ in the general argument $\xi_x^\nu$ is given.

## § 3. CONDITION THAT A FORM CONTAIN A GIVEN QUADRATIC FORM AS A FACTOR.

Using the expansion (6) obtained in § 2, a minimum set of necessary and sufficient conditions that a given form be divisible by the $r$th power of a quadratic form is the identical vanishing of $F$, $\frac{\partial F}{\partial \xi_2}$, $\frac{\partial^2 F}{\partial \xi_2^2}$, $\cdots$ $\frac{\partial^{r-1} F}{\partial \xi_2^{r-1}}$. Since these expressions are linear in $x$ and do not vanish identically for every value of $x$, then the identical vanishing of each of these $r$ expressions necessitates the vanishing of two seminvariants, and hence the simultaneous vanishing of $2r$ seminvariants furnishes the minimum set of necessary and sufficient conditions that $a_x^m$ be divisible by the $r$th power of $\xi_x^2$. In particular, if $a_x^m$ contain $\xi_x^2$ as a factor, then from (4) and (5)

$$(7)\qquad \begin{aligned} \alpha_1 &= a_m - \sum_{i=0}^{E(m-1/2)} \frac{\xi_2^{i+1}}{\lfloor i} \Omega_m^{i+2} a_{-\xi_1}^{m(i)} = 0, \\ \alpha_2 &= \sum_{i=0}^{E(m-1/2)} \frac{\xi_2^{i}}{\lfloor i} \Omega_m^{i+1} a_{-\xi_1}^{m(i)} = 0. \end{aligned}$$

and conversely. These two conditions are equivalent to the redundant set of $m+1$ conditions furnished by the identical vanishing of the covariant $\Phi$, given by Clebsch,* which for $m = 4$ is

$$\Phi = -2D(a\xi)\xi_x \cdot a_x^3 + 2\xi_x^2 (a\xi')(a\xi'')\xi_x' a_x = 0,$$

where $D$ is the discriminant of $\xi_x^2$ and $\xi_x^2 = \xi_x'^2 = \xi_x''^2$. B. Igel† has developed a method depending upon the vanishing of a series of resultants of every two forms of a certain system.

* Theorie der Binaren Formen, p. 94.

† *Sitzungsberichte der Kön. Akad. der Wissenshaften Mathematische.*, Bd. LXXXII, p. 943.

## § 4. SEPARATION OF $a_x^m/(\xi_x^2)^k$ INTO PARTIAL FRACTIONS.

From (6) § 2, we have

$$\frac{a_x^m}{(\xi_x^2)^k} = \frac{F - \dfrac{\partial F}{\partial \xi_2}\xi_x^2 + \dfrac{1}{\lfloor 2}\dfrac{\partial^2 F}{\partial \xi_2^2}(\xi_x^2)^2 + \cdots + \dfrac{(-1)^k}{\lfloor k}\dfrac{\partial^k F}{\partial \xi_2^k}(\xi_x^2)^k + \cdots \dfrac{(-1)^n}{\lfloor n}\dfrac{\partial^n F}{\partial \xi_2^n}(\xi_x^2)^n}{(\xi_x^2)^k}.$$

Hence

$$\frac{a_x^m}{(\xi_x^2)^k} = \frac{F}{(\xi_x^2)^k} - \frac{\dfrac{\partial F}{\partial \xi_2}}{(\xi_x^2)^{k-1}} + \cdots \frac{(-1)^{k-1}}{\lfloor k-1}\frac{\dfrac{\partial^{k-1}F}{\partial \xi_2^{k-1}}}{\xi_x^2} + \frac{(-1)^k}{\lfloor k}\cdot\frac{\partial^k F}{\partial^k \xi_2} + \frac{(-1)^{k+1}}{\lfloor k+1}\frac{\partial^{k+1}F}{\partial \xi_2^{k+1}}\cdot(\xi_x^2) + \cdots \frac{(-1)^n}{\lfloor n}\frac{\partial^n F}{\partial \xi_2^n}\cdot(\xi_x^2)^{n-k},$$

or, upon writing the numerator in explicit form,

$$\frac{a_x^m}{(\xi_x^2)^k} = \frac{a_m + \sum_{i=0}^{E[(m-1)/2]} - \dfrac{{\xi_2}^{i+1}}{\lfloor i}\Omega_m{}^{i+2}a_{-\xi_1}^{m(i)} + \dfrac{{\xi_2}^i}{\lfloor i}\Omega_m{}^{i+1}a_{-\xi_1}^{m(i)}x}{(\xi_x^2)^k} + \sum_{s=1}^{E(m/2)} \frac{\sum_{i=0}^{E[(m-1)/2]} \dfrac{(i+1){\xi_2}^{i-s+1}}{\lfloor i-s}\Omega_m{}^{i+2}a_{-\xi_1}^{m(i)} + \dfrac{{\xi_2}^{i-s}}{\lfloor i-s}\Omega_m{}^{i+1}a_{-\xi_1}^{m(i)}}{(\xi_x^2)^{k-s}}.$$

To illustrate, let

$$\frac{a_x^4}{(\xi_x^2)^3} = \frac{x^4 - x^3 + 3x^2 - x}{(x^2 - x + 1)^3} = \frac{F}{(x^2 - x + 1)^3} - \frac{\partial F/\partial \xi_2}{(x^2 - x + 1)^2} + \frac{\partial^2 F/\partial {\xi_2}^2}{\lfloor 2(x^2 - x + 1)}.$$

From (5)

$$F = -({\xi_1}^2 + \xi_1 + 3)\xi_2 + {\xi_2}^2 + \{(-{\xi_1}^3 - {\xi_1}^2 - 3\xi_1 - 1) + (2\xi_1 + 1)\xi_2\}x,$$

$$\frac{\partial F}{\partial \xi_2} = -(\xi_1^2 + \xi_1 + 3) + 2\xi_2 + (2\xi_1 + 1)x,$$

$$\frac{\partial^2 F}{\lfloor 2 \partial \xi_2^2} = 1.$$

Now, $\xi_1 = -1$, $\xi_2 = 1$.

$$\therefore \quad F = x - 2, \quad \frac{\partial F}{\partial \xi_2} = -x - 1, \quad \frac{1}{\lfloor 2} \frac{\partial^2 F}{\partial \xi_2^2} = 1.$$

Then

$$\frac{x^4 - x^3 + 3x^2 - x}{(x^2 - x + 1)^3} = \frac{x - 2}{(x^2 - x + 1)^3} + \frac{x + 1}{(x^2 - x + 1)^2} + \frac{1}{x^2 - x + 1}.$$

## § 5. COMPUTATION OF SYMMETRIC FUNCTIONS.

The two seminvariants obtained in § 3, whose identical vanishing furnished the necessary and sufficient conditions that a form of order $m$ be divisible by a given quadratic form, also furnish a direct method for the computation of certain classes of symmetric functions.

Let the $m$ roots of

$$f(x) = a_x^m = a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \cdots + a_m = 0$$

be $x_1, x_2, \cdots x_m$, and denote the totality of the sums of these roots taken in pairs by $\xi_{1i}$ and the products in pairs by $\xi_{2j}$. $[i, j = 1, 2, \cdots \frac{1}{2}m(m-1).]$

The number of quadratic forms which will divide $a_x^m$ is $\frac{1}{2}m(m-1)$ and they are of the form

$$x^2 - (x_i + x_j)x + x_i x_j = x^2 + \xi_{1k} x + \xi_{2k}. \tag{8}$$

Hence, if from the two conditions (7),

$$\alpha_1 = a_m - \sum_{i=0}^{E[(m-1)/2]} \frac{{\xi_2}^{i+1}}{\lfloor i} \Omega_m{}^{i+2} a_{-\xi_1}^{m(i)} = 0,$$

$$\alpha_2 = \sum_{i=0}^{E[(m-1)/2]} \frac{{\xi_2}^{i}}{\lfloor i} \Omega_m{}^{i+1} a_{-\xi_1}^{m(i)} = 0,$$

where $\alpha_1$ is of order $m-2$ in $\xi_1$ and $E\left(\frac{m+1}{2}\right)$ in $\xi_2$, and $\alpha_2$ is of order $m-1$ in $\xi_1$ and $E\left(\frac{m-1}{2}\right)$ in $\xi_2$, we eliminate $\xi_2$,

we obtain a resolvent of order $\frac{1}{2}m(m-1)$,

$$\Phi = \sum_{i=0}^{\frac{1}{2}m(m-1)} (-1)^i \phi_i \xi_1^{\frac{1}{2}m(m-1)-i} = 0, \tag{9}$$

whose roots are the $\xi_{1i}$. The $\phi_i$ are values of the elementary symmetric functions of the $\xi_{1i}$ and are rational functions of the $a$'s. By eliminating $\xi_1$, we obtain in the same manner

$$\Psi = \sum_{j=0}^{\frac{1}{2}m(m-1)} (-1)^j \psi_j \xi_2^{\frac{1}{2}m(m-1)-j} = 0, \tag{10}$$

a resolvent whose roots are the $\xi_{2j}$. The $\psi_j$ are the values of the elementary symmetric functions of $\xi_{2j}$ and are expressible rationally in terms of the $a$'s.

## § 6. THE RESOLVENTS OF KÖNIG.

The $\Psi = 0$ of § 5, is the König resolvent for the case $\nu = 2$. $\Phi = 0$ is a resolvent of a similar nature. Both of these resolvents have been studied by Glenn* in his theory of the rational resolvents of the factorable ternary forms.

The general form of the $\Phi$ resolvent is, from (7)

$$\Phi\dagger = \begin{vmatrix} \Omega a^m_{-\xi_1}, & \Omega^2 a^{m(1)}_{-\xi_1}, & \cdots & \Omega^{E(m+1/2)} a^{m[E(m-1)/2]}_{-\xi_1}, & 0, & \cdots & 0 \\ 0, & \Omega a^m_{-\xi_1}, & \Omega^2 a^{m(1)}_{-\xi_1}, & \cdots & \Omega^{E(m+1/2)} a^{m[E(m-1/2)]}_{-\xi_1}, & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_m, & \Omega^2 a^m_{-\xi_1}, & \Omega^3 a^{m(1)}_{-\xi_1}, & \cdots & \Omega^{E(m+3/2)} a^{m[E(m-1/2)]}_{-\xi_1}, & \cdots, & 0 \\ 0, & a_m, & \Omega^2 a^m_{-\xi_1}, & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \cdots & \cdots & \Omega^{E(m+3/2)} a^{m[E(m-1/2)]}_{-\xi_1} & & \end{vmatrix} = 0.$$

The elimination of $\xi_1$ from equations (7) does not exhibit the $\Psi\ddagger$ resolvent in a concise form like the determinant above.

The determination of the $\frac{1}{2}m(m-1)$ quadratic forms which divide a given form can be obtained by solving either $\Phi = 0$ or $\Psi = 0$. If $\Phi = 0$ be solved, then by substituting the $\xi_{1i}$ in the condition $\alpha_2$ of (3), $\xi_{2i}$ can be obtained and the quadratic factor is determined.

* *Amer. Jour. of Math.*, Vol. 32 (1910), p. 80.

† The order of the determinant is $m - 1$.

‡ The order of the $\Psi$ determinant is $m - 1 + m - 2 = 2m - 3$.

These resolvents are given for the case $m = 4$. From $(4^a)$,

$$\alpha_1 = -a_0\xi_2^2 + (a_0\xi_1^2 - a_1\xi_1 + a_2)\xi_2 + a_4 = 0,$$

$$\alpha_2 = (2a_0\xi_1 - a_1)\xi_2 + (-a_0\xi_1^3 + a_1\xi_1^2 - a_2\xi_1 + a_3) = 0.$$

Eliminating $\xi_2$,

$$\Phi = \begin{vmatrix} -a_0, & a_0\xi_1^2 - a_1\xi_1 + a_2, & a_4 \\ 2a_0\xi_1 - a_1, & -a_0\xi_1^3 + a_1\xi_1^2 - a_2\xi_1 + a_3, & 0 \\ 0, & 2a_0\xi_1 - a_1, & -a_0\xi_1^3 + a_1\xi_1^2 - a_2\xi_1 + a_3 \end{vmatrix} = 0.$$

Likewise, eliminating $\xi_1$,

$$\Psi = \begin{vmatrix} a_0\xi_2, & -a_1\xi_2, & -a_0\xi_2^2 + a_2\xi_2 + a_4, & 0 & 0 \\ 0, & a_0\xi_2, & -a_1\xi_2, & -a_0\xi_2^2 + a_2\xi_2 + a^4, & 0 \\ 0, & 0, & a_0\xi_2, & -a_1\xi_2, & -a_0\xi_2^2 + a_2\xi_2 + a_4 \\ -a_0, & a_1, & 2a_0\xi_2 - a_2, & -a_1\xi_2 + a_3, & 0 \\ 0, & -a_0, & a_1, & 2a_0\xi_2 - a_2, & -a_1\xi_2 + a_3 \end{vmatrix} = 0.$$

The explicit expanded forms of $\Phi = 0$ and $\Psi = 0$ can be obtained from the tables in § 7.

## § 7. TABLES OF SYMMETRIC FUNCTIONS OF THE SUMS AND OF THE PRODUCTS OF THE ROOTS OF A GIVEN FORM, TAKEN IN PAIRS.

In § 5, it was shown that there existed two resolvents $\Phi = 0$ and $\Psi = 0$, whose roots were the $\xi_{1i}$ and the $\xi_{2j}$ of (8) respectively, and hence the coefficients of the powers of $\xi_1$ in $\Phi = 0$ and $\xi_2$ in $\Psi = 0$ are the values of the elementary symmetric functions of $\xi_{1i}$ and $\xi_{2j}$, respectively, expressed rationally in terms of the $a$'s of $f = a_x^m$. These symmetric functions have been computed

for the cases $m = 3, 4, 5, 6$ and a table of them in homogeneous form completes this paper. In order to check these results, the following formulas have been derived:

(a)
$$\phi_k = \sum_i \lambda_i {a_0}^{p_i} {a_1}^{q_i} \cdots {a_m}^{t_i},$$
$$(-2)^k \binom{{}_mC_2}{k} = \sum_i \lambda_i \binom{m}{0}^{p_i} \binom{m}{1}^{q_i} \cdots \binom{m}{m}^{t_i}.$$

(b)
$$\psi_k = \sum_i \mu_i {a_0}^{p_i} {a_1}^{q_i} \cdots {a_m}^{t_i},$$
$$\binom{{}_mC_2}{k} = \sum_i \mu_i \binom{m}{0}^{p_i} \binom{m}{1}^{q_i} \cdots \binom{m}{m}^{t_i}.$$

As a further check upon the $\psi$'s, it may be noticed that by interchanging $a_0$ with $a_m$, $a_1$ with $a_{m-1}$, etc., $\psi_j$ is transformed into $\psi_{m-j}$. $(j = 0, 1, 2, \cdots m.)$

Table Ia.* $m = 3.$

$$\phi_1 = 2a_0a_1,$$
$$\phi_2 = {a_1}^2 + a_0a_2,$$
$$\phi_3 = a_1a_2 - a_0a_3.$$

Table Ib.

$$\psi_1 = a_0a_2,$$
$$\psi_2 = a_1a_3,$$
$$\psi_3 = {a_3}^2.$$

TABLE IIa. $m = 4.$

| $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ | $\phi_5$ | $\phi_6$ |
|---|---|---|---|---|---|
| $3{a_0}^2a_1$ | $2{a_0}^2a_2$<br>$+ 3a_0{a_1}^2$ | $4a_0a_1a_2$<br>$+ {a_1}^3$ | $- 4{a_0}^2a_4$<br>$+ a_0a_1a_3$<br>$+ a_0{a_2}^2$<br>$+ 2{a_1}^2a_2$ | $- 4a_0a_1a_4$<br>$+ a_1{a_2}^2$ | $- a_0{a_3}^2$<br>$- {a_1}^2a_4$<br>$+ a_1a_2a_3$ |

TABLE IIb.

| $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ | $\psi_5$ | $\psi_6$ |
|---|---|---|---|---|---|
| ${a_0}^2a_2$ | $- {a_0}^2a_4$<br>$+ a_0a_1a_3$ | $- 2a_0a_2a_4$<br>$+ a_0{a_3}^2$<br>$+ {a_1}^2a_4$ | $- a_0{a_4}^2$<br>$+ a_1a_3a_4$ | $+ a_2{a_4}^2$ | ${a_4}^3$ |

* $\phi_0 = \psi_0 = {a_0}^{m-1}.$

TABLE IIIa. $m = 5$.

| $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ | $\phi_5$ |
|---|---|---|---|---|
| $4a_0^3a_1$ | $3a_0^3a_2$<br>$+ 6a_0^2a_1^2$ | $a_0^3a_3$<br>$+ 9a_0^2a_1a_2$<br>$+ 4a_0a_1^3$ | $- 3a_0^3a_4$<br>$+ 4a_0^2a_1a_3$<br>$+ 3a_0^2a_2^2$<br>$+ 9a_0a_1^2a_2$<br>$+ a_1^4$ | $- 11a_0^3a_5$<br>$- 5a_0^2a_1a_4$<br>$+ 2a_0^2a_2a_3$<br>$+ 5a_0a_1^2a_3$<br>$+ 6a_0a_1a_2^2$<br>$+ 3a_1^3a_2$ |

TABLE IIIa (*continued*).

| $\phi_6$ | $\phi_7$ | $\phi_8$ | $\phi_9$ | $\phi_{10}$ |
|---|---|---|---|---|
| $- 22a_0^2a_1a_5$<br>$- 2a_0^2a_2a_4$<br>$- a_0^2a_3^2$<br>$- 2a_0a_1^2a_4$<br>$+ 6a_0a_1a_2a_3$<br>$+ a_0a_2^3$<br>$+ 2a_1^3a_3$<br>$+ 3a_1^2a_2^2$ | $- 4a_0^2a_2a_5$<br>$- 4a_0^2a_3a_4$<br>$- 16a_0a_1^2a_5$<br>$+ a_0a_2^2a_3$<br>$+ 4a_1^2a_2a_3$<br>$+ a_1a_2^3$ | $7a_0^2a_3a_5$<br>$- 4a_0^2a_4^2$<br>$- 9a_0a_1a_2a_5$<br>$- 3a_0a_1a_3a_4$<br>$+ a_0a_2^2a_4$<br>$- a_0a_2a_3^2$<br>$- 4a_1^3a_5$<br>$+ a_1^2a_2a_4$<br>$+ a_1^2a_3^2$<br>$+ 2a_1a_2^2a_3$ | $4a_0^2a_4a_5$<br>$+ 4a_0a_1a_3a_5$<br>$- 4a_0a_1a_4^2$<br>$- a_0a_2^2a_5$<br>$- a_0a_3^3$<br>$- 4a_1^2a_2a_5$<br>$- a_1a_2^2a_4$<br>$+ a_1a_2a_3^2$ | $- a_0^2a_5^2$<br>$+ 2a_0a_1a_4a_5$<br>$+ a_0a_2a_3a_5$<br>$- a_0a_3^2a_4$<br>$- a_1^2a_4^2$<br>$- a_1a_2^2a_5$<br>$+ a_1a_2a_3a_4$ |

TABLE IIIb.

| $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ | $\psi_5$ |
|---|---|---|---|---|
| $a_0^3a_2$ | $- a_0^3a_4$<br>$+ a_0^2a_1a_3$ | $- a_0^2a_1a_5$<br>$- 2a_0^2a_2a_4$<br>$+ a_0^2a_3^2$<br>$+ a_0a_1^2a_4$ | $+ a_0^2a_3a_5$<br>$- a_0^2a_4^2$<br>$- 3a_0a_1a_2a_5$<br>$+ a_0a_1a_3a_4$<br>$+ a_1^3a_5$ | $2a_0^2a_5^2$<br>$- 2a_0a_1a_4a_5$<br>$- 2a_0a_2a_3a_5$<br>$+ a_0a_2a_4^2$<br>$+ a_1^2a_3a_5$ |

TABLE IIIb (*continued*).

| $\psi_6$ | $\psi_7$ | $\psi_8$ | $\psi_9$ | $\psi_{10}$ |
|---|---|---|---|---|
| $a_0a_2a_5^2$<br>$- 3a_0a_3a_4a_5$<br>$+ a_0a_4^3$<br>$- a_1^2a_5^2$<br>$+ a_1a_2a_4a_5$ | $- a_0a_4a_5^2$<br>$- 2a_1a_3a_5^2$<br>$+ a_1a_4^2a_5$<br>$+ a_2^2a_5^2$ | $- a_1a_5^3$<br>$+ a_2a_4a_5^2$ | $a_3a_5^3$ | $a_5^4$ |

TABLE IVa. $m = 6$.

| $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ | $\phi_5$ | $\phi_6$ | $\phi_7$ | $\phi_8$ |
|---|---|---|---|---|---|---|---|
| $5a_0^4a_1$ | $4a_0^4a_2$<br>$+ 10a_0^3a_1^2$ | $2a_0^4a_3$<br>$+ 16a_0^3a_1a_2$<br>$+ 10a_0^2a_1^3$ | $- 2a_0^4a_4$<br>$+ 9a_0^3a_1a_3$<br>$+ 6a_0^3a_2^2$<br>$+ 24a_0^2a_1^2a_2$<br>$+ 5a_0a_1^4$ | $- 10a_0^4a_5$<br>$- 4a_0^3a_1a_4$<br>$+ 6a_0^3a_2a_3$<br>$+ 15a_0^2a_1^2a_3$<br>$+ 18a_0^2a_1a_2^2$<br>$+ 16a_0a_1^3a_2$<br>$+ a_1^5$ | $- 26a_0^4a_6$<br>$- 29a_0^3a_1a_5$<br>$- 2a_0^3a_2a_4$<br>$- a_0^2a_1^2a_4$<br>$+ 21a_0^2a_1a_2a_3$<br>$+ 4a_0^2a_2^3$<br>$+ 11a_0a_1^3a_3$<br>$+ 18a_0a_1^2a_2^2$<br>$+ 4a_1^4a_2$ | $- 78a_0^3a_1a_6$<br>$- 12a_0^3a_2a_5$<br>$- 6a_0^3a_3a_4$<br>$- 32a_0^2a_1^2a_5$<br>$+ 3a_0^2a_1a_3^2$<br>$+ 2a_0^2a_1a_2a_4$<br>$+ 6a_0^2a_2^2a_3$<br>$+ 2a_0a_1^3a_4$<br>$+ 24a_0a_1^2a_2a_3$<br>$+ 8a_0a_1a_2^3$<br>$+ 3a_1^4a_3$<br>$+ 6a_1^3a_2^2$ | $- 24a_0^3a_2a_6$<br>$- 3a_0^3a_3a_5$<br>$- 7a_0^3a_4^2$<br>$- 94a_0^2a_1^2a_6$<br>$- 26a_0^2a_1a_2a_5$<br>$- 8a_0^2a_1a_3a_4$<br>$+ 2a_0^2a_2^2a_4$<br>$- 16a_0a_1^3a_5$<br>$+ 8a_0a_1^2a_2a_4$<br>$+ 6a_0a_1^2a_3^2$<br>$+ 15a_0a_1a_2^2a_3$<br>$+ a_0a_4^2$<br>$+ a_1^4a_4$<br>$+ 9a_1^3a_2a_3$<br>$+ 4a_1^2a_2^3$ |

TABLE IVa (*continued*).

| $\phi_9$ | $\phi_{10}$ | $\phi_{11}$ | $\phi_{12}$ | $\phi_{13}$ | $\phi_{14}$ | $\phi_{15}$ |
|---|---|---|---|---|---|---|
| $3a_0^3a_3a_6$ | $54a_0^3a_4a_6$ | $18a_0^3a_5a_6$ | $-27a_0^3a_6^2$ | $-27a_0^2a_1a_6^2$ | $-9a_0a_3a_5a_6$ | $-a_0^2a_5^3$ |
| $-10a_0^3a_4a_5$ | $-12a_0^3a_5^2$ | $+84a_0^2a_1a_4a_6$ | $+33a_0^2a_1a_5a_6$ | $+6a_0^2a_3a_5^2$ | $+3a_0^2a_4a_5^2$ | $+3a_0a_1a_3a_5a_6$ |
| $-76a_0^2a_1a_2a_6$ | $+33a_0^2a_1a_3a_6$ | $-23a_0^2a_1a_5^2$ | $+18a_0^2a_2a_4a_6$ | $+21a_0a_1^2a_5a_6$ | $-9a_0a_1^2a_6^2$ | $+2a_0a_1a_4a_5^2$ |
| $-7a_0^2a_1a_3a_5$ | $-21a_0^2a_1a_4a_5$ | $+6a_0^2a_2a_3a_6$ | $-5a_0^2a_2a_5^2$ | $+18a_0a_1a_2a_4a_6$ | $+6a_0a_1a_2a_5a_6$ | $+a_0a_2a_3a_5^2$ |
| $-13a_0^2a_1a_4^2$ | $-18a_0^2a_2^2a_6$ | $-2a_0^2a_2a_4a_5$ | $+3a_0^2a_3^2a_6$ | $-9a_0a_1a_2a_5^2$ | $+3a_0a_1a_3a_4a_6$ | $-a_0a_3^3a_6$ |
| $-2a_0^2a_2^2a_5$ | $+6a_0^2a_2a_3a_5$ | $+6a_0^2a_3^2a_5$ | $+7a_0^2a_3a_4a_5$ | $+3a_0a_1a_3^2a_6$ | $+4a_0a_1a_3a_5^2$ | $-a_0a_3^2a_4a_5$ |
| $-4a_0^2a_2a_3a_4$ | $-6a_0^2a_2a_4^2$ | $-8a_0^2a_3a_4^2$ | $-4a_0^2a_4^3$ | $+3a_0a_1a_3a_4a_5$ | $-2a_0a_1a_4^2a_5$ | $-a_1^3a_6^2$ |
| $-2a_0^2a_3^3$ | $-6a_0^2a_3^2a_4$ | $+4a_0a_1^2a_3a_6$ | $+43a_0a_1^2a_4a_6$ | $-4a_0a_1a_4^3$ | $-a_0a_2^2a_5^2$ | $-2a_1^2a_2a_5a_6$ |
| $-52a_0a_1^3a_6$ | $-72a_0a_1^2a_2a_6$ | $-16a_0a_1^2a_4a_5$ | $-16a_0a_1^2a_5^2$ | $-2a_0a_3^3a_4$ | $+3a_0a_2a_3^2a_6$ | $-a_1^2a_3a_4a_6$ |
| $-20a_0a_1^2a_2a_5$ | $-7a_0a_1^2a_3a_5$ | $-34a_0a_1a_2^2a_6$ | $-5a_0a_1a_2a_3a_6$ | $+7a_1^3a_4a_6$ | $+a_0a_2a_3a_4a_5$ | $-a_1^2a_4^2a_5$ |
| $+8a_0a_1a_2^2a_4$ | $-7a_0a_1^2a_4^2$ | $+2a_0a_1a_2a_3a_5$ | $-7a_0a_1a_2a_4a_5$ | $-4a_1^3a_5^2$ | $-a_0a_3^3a_5$ | $-a_1a_2^2a_5^2$ |
| $-6a_0a_1a_2a_3^2$ | $-5a_0a_1a_2^2a_5$ | $-4a_0a_1a_2a_4^2$ | $+4a_0a_1a_3^2a_5$ | $-9a_1^2a_2a_3a_6$ | $-a_0a_3^2a_4^2$ | $+a_1a_2a_3^2a_6$ |
| $+2a_0a_2^3a_3$ | $+4a_0a_1a_2a_3a_4$ | $-4a_0a_1a_3^2a_4$ | $-7a_0a_1a_3a_4^2$ | $-3a_1^2a_2a_4a_5$ | $+4a_1^3a_5a_6$ | $+a_1a_2a_3a_4a_5$ |
| $-3a_1^4a_5$ | $-a_0a_1a_3^3$ | $+2a_0a_2^2a_3a_4$ | $-4a_0a_2^3a_6$ | $+a_1^2a_3^2a_5$ | $+4a_1^2a_2a_4a_6$ | |
| $+4a_1^3a_2a_4$ | $+2a_0a_2^3a_4$ | $-2a_0a_2a_3^3$ | $+a_0a_2^2a_3a_5$ | $-a_1^2a_3a_4^2$ | $-4a_1^2a_2a_5^2$ | |
| $+3a_1^3a_3^2$ | $-11a_1^4a_6$ | $-22a_1^3a_2a_6$ | $+a_0a_2^2a_4^2$ | $-4a_1a_2^3a_6$ | $-a_1^2a_3^2a_6$ | |
| $+9a_1^2a_2^2a_3$ | $-5a_1^3a_2a_5$ | $-2a_1^3a_3a_5$ | $-2a_0a_2a_3^2a_4$ | $+a_1a_2^2a_3a_5$ | $-a_1^2a_4^3$ | |
| $+a_1a_2^4$ | $+2a_1^3a_3a_4$ | $-a_1^3a_4^2$ | $-a_0a_3^4$ | $+a_1a_2^2a_4^2$ | $-4a_1a_2^2a_3a_6$ | |
| | $+5a_1^2a_2^2a_4$ | $-2a_1^2a_2^2a_5$ | $-4a_1^3a_3a_6$ | $+2a_1a_2a_3^2a_4$ | $+a_1a_2a_3^2a_5$ | |
| | $+6a_1^2a_2a_3^2$ | $+6a_1^2a_2a_3a_4$ | $-4a_1^3a_4a_5$ | | $+a_1a_2a_3a_4^2$ | |
| | $+3a_1a_2^3a_3$ | $+a_1^2a_3^3$ | $-16a_1^2a_2^2a_6$ | | | |
| | | $+2a_1a_2^3a_4$ | $+a_1^2a_3^2a_4$ | | | |
| | | $+3a_1a_2^2a_3^2$ | $+4a_1a_2^2a_3a_4$ | | | |
| | | | $+a_1a_2a_3^3$ | | | |

TABLE IVb.

| $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ | $\psi_5$ | $\psi_6$ | $\psi_7$ | $\psi_8$ |
|---|---|---|---|---|---|---|---|
| $a_0^4a_2$ | $-a_0^4a_4$ | $a_0^4a_6$ | $2a_0^3a_2a_6$ | $-2a_0^3a_4a_6$ | $-2a_0^2a_6^2$ | $+3a_0^2a_3a_5a_6$ | $+a_0^2a_5^2a_6$ |
| | $+a_0^3a_1a_2$ | $-2a_0^3a_1a_5$ | $+a_0^3a_3a_5$ | $+2a_0^3a_5^2$ | $+3a_0^2a_1a_5a_6$ | $+a_0^2a_4^2a_6$ | $+3a_0a_1a_3a_6^2$ |
| | | $-2a_0^3a_2a_4$ | $-a_0^3a_4^2$ | $+3a_0^2a_1a_3a_6$ | $+a_0^2a_2a_5^2$ | $-a_0^2a_4a_5^2$ | $+a_0a_1a_4a_5a_6$ |
| | | $+a_0^3a_3^2$ | $-a_0^2a_1^2a_6$ | $-2a_0^2a_1a_4a_5$ | $+3a_0^2a_3^2a_6$ | $+a_0a_1^2a_6^2$ | $-a_0a_1a_5^3$ |
| | | $+a_0^2a_1^2a_4$ | $-3a_0^2a_1a_2a_5$ | $+2a_0^2a_2^2a_6$ | $-3a_0^2a_3a_4a_5$ | $+a_0a_1a_2a_5a_6$ | $+a_0a_2^2a_6^2$ |
| | | | $+a_0^2a_1a_3a_4$ | $-2a_0^2a_2a_3a_5$ | $+a_0^2a_4^3$ | $-a_0a_1a_3a_4a_6$ | $-a_0a_2a_3a_5a_6$ |
| | | | $+a_0^2a_2a_3^2$ | $+a_0^2a_2a_4^2$ | $-a_0a_1^2a_4a_6$ | $-2a_0a_1a_3a_5^2$ | $-2a_0a_2a_4^2a_6$ |
| | | | $+a_0a_1^3a_5$ | $-4a_0a_1^2a_2a_6$ | $-a_0a_1^2a_5^2$ | $+a_0a_1a_4^2a_5$ | $+a_0a_2a_4a_5^2$ |
| | | | | $+a_0a_1^2a_3a_5$ | $-3a_0a_1a_2a_5a_6$ | $-2a_0a_2^2a_4a_6$ | $-a_1^2a_2a_6^2$ |
| | | | | $+a_1^4a_6$ | $+a_0a_1a_2a_4a_5$ | $+a_0a_2^2a_5^2$ | $-2a_1^2a_3a_5a_6$ |
| | | | | | $+a_1^3a_3a_6$ | $-a_1^3a_5a_6$ | $+a_1^2a_4^2a_6$ |
| | | | | | | $+a_1^2a_2a_4a_6$ | $+a_1a_2^2a_5a_6$ |

TABLE IVb (*continued*).

| $\psi_9$ | $\psi_{10}$ | $\psi_{11}$ | $\psi_{12}$ | $\psi_{13}$ | $\psi_{14}$ | $\psi_{15}$ |
|---|---|---|---|---|---|---|
| $-2a_0^2a_6^3$ | $-2a_0a_2a_6^3$ | $+2a_0a_4a_6^3$ | $+a_0a_6^4$ | $-a_2a_6^4$ | $+a_4a_6^4$ | $a_6^5$ |
| $+3a_0a_1a_5a_6^2$ | $+3a_0a_3a_5a_6^2$ | $-a_0a_5^2a_6^2$ | $-a_1a_5a_6^3$ | $+a_3a_5a_6^3$ | | |
| $-a_0a_2a_5^2a_6$ | $+2a_0a_4^2a_6^2$ | $+a_1a_3a_6^3$ | $-2a_2a_4a_6^3$ | | | |
| $+3a_0a_3^2a_6^2$ | $-4a_0a_4a_5^2a_6$ | $-3a_1a_4a_5a_6^2$ | $+a_2a_5^2a_6^2$ | | | |
| $-3a_0a_3a_4a_5a_6$ | $+a_0a_5^4$ | $+a_1a_5^3a_6$ | $+a_3^2a_6^3$ | | | |
| $+a_0a_3a_5^3$ | $+2a_1^2a_6^3$ | $-a_2^2a_6^3$ | | | | |
| $+a_1^2a_4a_6^2$ | $-2a_1a_2a_5a_6^2$ | $+a_2a_3a_5a_6^2$ | | | | |
| $-a_1^2a_5^2a_6$ | $-2a_1a_3a_4a_6^2$ | | | | | |
| $-3a_1a_2a_3a_6^2$ | $+a_1a_3a_5^2a_6$ | | | | | |
| $+a_1a_2a_4a_5a_6$ | $+a_2^2a_4a_6^2$ | | | | | |
| $+a_2^3a_6^2$ | | | | | | |

UNIVERSITY OF PENNSYLVANIA

# THE RESOLVENTS OF KÖNIG AND OTHER TYPES OF SYMMETRIC FUNCTIONS

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STANLEY P. SHUGERT

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# THE RESOLVENTS OF KÖNIG AND OTHER TYPES OF SYMMETRIC FUNCTIONS.

By Stanley P. Shugert.

## INTRODUCTION.

Symmetric functions of the differences of the roots of an equation are an important part of many theories, particularly of binary invariant theory, and much attention has been given to the computation of such functions; while the functions which are not symmetric in differences have received very little attention, but in some instances have an important bearing in general theories. As a particular instance, there are the resolvents of degree $\binom{m}{r}$ of an equation,

$$f(x) = x^m + A_1x^{m-1} + A_2x^{m-2} + \cdots + A_m = 0,$$

which were studied by J. König in the *Mathematische Annalen*, Vol. 15 (1879). In brief, these are constructed as follows: the roots of

$$F(z) = f(z + \mu) = z^m + B_1z^{m-1} + \cdots + B_m = 0$$

are $(\alpha_i - \mu)$ $\{i = 1, 2, \cdots m\}$, where the $\alpha_i$ are the roots of $f(x) = 0$. The equation whose roots are the $\binom{m}{r}$ quantities of the type,

$$(-1)^r(\alpha_1 - \mu)(\alpha_2 - \mu) \cdots (\alpha_r - \mu) \quad (r = 1, 2, \cdots m),$$

is rational in $B_i$ and is a resolvent of $f(x) = 0$.

In volume 15 (1914) of the *Transactions of the American Mathematical Society*, Prof. O. E. Glenn has published a paper in which is given the expansion of the binary form of order $m = \mu\nu$,

$$f(x) = a_0x_1^m + a_1x_1^{m-1}x_2 + \cdots + a_mx_2^m$$

as a power series in which the argument is the binary form of order $\nu$,

$$\xi_x^\nu = \xi_0x_1^\nu + \xi_1x_1^{\nu-1}x_2 + \cdots + \xi_\nu x_2^\nu.$$

In § 2 of this paper, we obtain an independent proof of his result

for $\nu = 2$ by a method depending upon the solution of a system of linear equations and a preliminary lemma on determinants, § 1. The remainder of the present paper is devoted to a succession of applications, introducing theories, which depend upon this expansion for the case $\nu = 2$, and finally introducing in § 6 a method for the determination of the König Resolvents with the aid of the tables given in § 7. In § 3 a minimum set of seminvariant conditions that a given quadratic form may divide a given form $f$ is obtained and § 4 is devoted to a problem in rational fractions. There are obtained in § 5, methods for the computation of symmetric functions of the sums and of the products of the roots of an equation, taken in pairs.

## § 1. A LEMMA ON DETERMINANTS.

We first prove the following lemma concerning determinants:

LEMMA: *The determinant* $\{a_{rs}\}$ *in which every constituent is zero for* $s > r + 1$ *and not zero for* $s = r$ *can be reduced to a determinant* $\{a_{rs}\}$ *in which every constituent is zero for* $s > r$.

Let $\{a_{rs}\} =$

$$\text{(A)} \qquad \begin{vmatrix} a_{11}, & a_{12}, & 0 & \cdots & 0, & 0 & \cdots & 0 \\ a_{21}, & a_{22}, & a_{23}, & \cdots & 0, & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & \\ a_{i1}, & a_{i2}, & \cdots & \cdots & a_{i\,i+1}, & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & 0 \\ a_{r-1\,1}, & a_{r-1\,2}, & a_{r-1\,3}, & \cdots & a_{r-1\,i+1}, & & \cdots & a_{r-1\,r} \\ a_{r\,1} & a_{r\,2}, & a_{r\,3}, & \cdots & a_{r\,i+1}, & & \cdots & a_{r\,r} \end{vmatrix}.$$

Multiply the last row by $a_{r-1\,r}/a_{r\,r}$ and subtract from the preceding row. This will give

$$\text{(B)} \qquad \begin{vmatrix} a_{11}, & a_{21} & 0, & \cdots & 0, & 0 & \cdots & 0 \\ a_{21}, & a_{22} & a_{23}, & \cdots & 0, & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ a_{i1}, & a_{i2} & \cdots & \cdots & a_{i\,i+1}, & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \dfrac{(a_{r-1}(a_r)_r)_1}{a_{rr}}, & \dfrac{(a_{r-1}(a_r)_r)_2}{a_{rr}}, & \cdots & \dfrac{(a_{r-1}(a_r)_r)_{i+1}}{a_{rr}}, & \cdots & \dfrac{(a_{r-1}(a_r)_r)_{r-1}}{a_{rr}}, & & 0 \\ a_{r1}, & a_{r2}, & \cdots & a_{r\,i+1}, & \cdots & a_{r\,r-1}, & & a_{rr} \end{vmatrix}$$

in which $(a_{r-1}(a_r)_r)_{i+1}$ is a symbolic notation for the second order

determinant $(a_{r-1\ i+1}a_{rr} - a_{r-1\ r}a_{r\ i+1})$. The minor determinant $A_{rr}$ is of the same type as the determinant $A$ and by applying the same process to it and continuing the method for each succeeding minor determinant, we finally obtain:

$$\begin{vmatrix} \dfrac{(a_1(a_2(a_3 \cdots (a_r)_r \cdots)_3)_2)_1}{a_2(a_3 \cdots (a_r)_r \cdots)_3)_2}, & 0, & \cdots 0 \\ \dfrac{(a_2(a_3(a_4 \cdots (a_r)_r \cdots)_4)_3)_1}{(a_3(a_4 \cdots (a_r)_r \cdots)_4)_3}, & \dfrac{(a_2(a_3 \cdots (a_r)_r \cdots)_3)_2}{(a_3(a_4 \cdots (a_r)_r \cdots)_4)_3}, & \cdots 0 \\ \cdots & \cdots & \cdots \\ \dfrac{a_i(a_{i+1} \cdots (a_r)_r \cdots)_{i+1})_1}{(a_{i+1}(a_{i+2} \cdots (a_r)_r \cdots)_{i+2})_{i+1}}, & \dfrac{(a_i(a_{i+1} \cdots (a_r)_r \cdots)_{i+1})_2}{a_{i+1}(a_{i+2} \cdots (a_r)_r \cdots)_{i+2})_{i+1}}, & \cdots \dfrac{a_i(a_{i+1} \cdots (a_r)_r \cdots)_{i+1})_j}{(a_{i+1}(a_{i+2} \cdots (a_r)_r \cdots)_{i+2})_{i+1}}, \cdots 0 \\ \cdots & \cdots & \cdots \\ \dfrac{(a_{r-1}(a_r)_r)_1}{a_{rr}}, & \dfrac{a_{r-1}(a_r)_r)_2}{a_{rr}}, & \cdots 0 \\ a_{r1}, & a_{r2}, & \cdots a_{rr} \end{vmatrix}$$

$(i = 1, 2, \cdots r,\ j = 1, 2, \cdots i)$.

The interlacing determinant notation is merely an extension of the symbolic method explained above for a determinant of the second order. These determinants are expanded by means of a succession of second order determinants and the method as shown for $(a_{r-1}(a_r)_r)_{i+1} = (a_{r-1\ i+1}a_{rr} - a_{r-ir}a_{ri+1})$ is to take the determinant subscripts, those written outside, first in an order symmetric with respect to the center with the product of the constituents and then to subtract the product obtained by reversing the determinant subscripts. Except in the last reduction, two constituents of the second order determinants will be symbolic determinants.

COROLLARY: *The determinant* $\{a_{rs}\}_{s \geqslant r+1}$ *can be written as a symbolic determinant: i. e.*

$$\{a_{rs}\}_{s \geqslant r+1} = (a_1(a_2(a_3(a_4 \cdots (a_r)_r \cdots)_4)_3)_2)_1.$$

## § 2. EXPANSION OF A POLYNOMIAL AS A POWER SERIES OF A QUADRATIC POLYNOMIAL.

THEOREM: *Any form of degree m can be expressed in terms of a power series of a quadratic form with linear expressions for the coefficients.*

First: if $m$ is even, $m = 2n$, we may write,

$$a_x^m = a_0x^{2n} + a_1x^{2n-1} + a_2x^{2n-2} + \cdots + a_{2n} = (\alpha_1 + \alpha_2x)$$

$$(1) \qquad + (\alpha_3 + \alpha_4x)\xi_x^2 + \cdots + (\alpha_{2r-1} + \alpha_{2r}x)(\xi_x^2)^{r-1} +$$

$$\cdots + (\alpha_{2n-1} + \alpha_{2n}x)(\xi_x^2)^{n-1} + (\xi_x^2)^n,$$

where $\xi_x^2 = \xi_0x^2 + \xi_1x + \xi_2$.

Since $a_0 = \xi_0^n$ and is determined, the number of $\alpha$'s on the right is equal to the number of $a$'s on the left and since the coefficients of the powers of $x$ on the right are linear in the $\alpha$'s, we have upon equating coefficients $2n = m$ linear non-homogeneous equations in the $2n$ quantities $\alpha_i$.

For the case $2n = 4$, we have

$$(a_0a_1a_2a_3a_4)(x)^4 = (\alpha_1 + \alpha_2x) + (\alpha_3 + \alpha_4x)(\xi_0x^2 + \xi_1x + \xi_2) + (\xi_0x^2 + \xi_1x + \xi_2)^2.$$

and upon multiplying out and rearranging,

$$\begin{aligned} \xi_0\alpha_4 &= a_1 - 2\xi_0\xi_1, \\ \xi_0\alpha_3 + \xi_1\alpha_4 &= a_2 - \xi_1^2 - 2\xi_0\xi_2, \\ \alpha_2 + \xi_1\alpha_3 + \xi_2\alpha_4 &= a_3 - 2\xi_1\xi_2, \\ \alpha_1 + \xi_2\alpha_3 &= a_4 - \xi_2^2. \end{aligned}$$

Second: If $m$ is odd $m = 2n - 1$, we can write

$$a_x^m = a_0x^{2n-1} + a_1x^{2n-2} + a_2x^{2n-3} + \cdots + a_{2n-1}$$

$$(1a) \qquad = (\alpha_1 + \alpha_2x) + (\alpha_3 + \alpha_4x)\xi_x^2 + \cdots$$

$$+ (\alpha_{2r-1} + \alpha_{2r}x)(\xi_x^2)^{r-1} + \cdots + (\alpha_{2n-1} + kx)(\xi_x^2)^{n-1},$$

where $k$ is determined to be $a_0/\xi_0^{n-1}$ and we have upon equating

coefficients $2n-1$ linear non-homogeneous equations to determine the $2n-1$ $\alpha$'s.

For the case $m=3$, we have

$$(a_0a_1a_2a_3)(x)^3 = (\alpha_1+\alpha_2x)+(\alpha_3+kx)(\xi_0x^2+\xi_1x+\xi_2)$$

and upon multiplying out and rearranging,

$$\begin{aligned}
\xi_0\alpha_3 &= a_1-\xi_1,\\
\alpha_2+\xi_1\alpha_3 &= a_2-\xi_2,\\
\alpha_1+\quad\ \xi_2\alpha_3 &= a_3,
\end{aligned}$$

if, as before, we take $a_0=\xi_0^{n-1}$, which we may do without loss of generality.

The particular form that these equations take, enable us to express them in general in terms of an operator, when we observe that any power of a form is a covariant of that form and so is given by its end term and a partial derivative operator. Hence the equations:

$$D^0\xi_0^{E[(m-1)/2]}\alpha_m = a_1 - D\xi_0^{E(m/2)},$$

$$D^0\xi_0^{E[(m-2)/2]}\alpha_{m-1} + D\xi_0^{E[(m-1)/2]}\alpha_m = a_2 - D^2\xi_0^{E(m/2)},$$

. . . . . .

$$\cdots + D^{i-j}\xi_0^{E[(m-j)/2]}\alpha_{m-j+1}\cdots$$

$$+ D^{i-2}\xi_0^{E[(m-2)/2]}\alpha_{m-1} + D^{i-1}\xi_0^{E[(m-1)/2]}\alpha_m = a_i - D^i\xi_0^{E(m/2)},$$

. . . . . .

$$D^0\xi_0{}^0\alpha_2 + \cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$D^0\xi_0{}^0\alpha_1 + D\xi_0{}^0\alpha_2\cdots + D^{i-j}\xi_0^{E[(m-j)/2]}\alpha_{m-j+1}\cdots$$

$$+ D^{m-2}\xi_0^{E[(m-2)/2]}\alpha_{m-1} + D^{m-1}\xi_0^{E[(m-1)/2]}\alpha_m = a_m - D^m\xi_0^{E(m/2)}$$

$(i=1, 2, 3, \cdots m)$ and $(j=1, 2, 3, \cdots m)$.

Here $E\left(\frac{m-j}{2}\right)$ is the integral part of $\frac{m-j}{2}$, and $D$ is the Aronhold operator

$$\left(D = \xi_1\frac{\partial}{\partial\xi_0} + \xi_2\frac{\partial}{\partial\xi_1}\right).$$

Also $D^{i-j}\xi_0^{E[(m-1)/2]} = 0$ for $j > i$.

Solving for $\alpha_{m-j+1}$, we have

$$\alpha_{m-j+1} =$$

$$\left|\begin{array}{cccccc}
0, & 0, & \cdots & \cdots, & a_1 - D\xi_0^{E(m/2)}, \\
0, & 0, & \cdots & \cdots, & a_2 - D^2\xi_0^{E(m/2)}, \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0, & 0, & \cdots D^{i-(j+1)}\xi_0^{E[(m-(j+1))/2]}, & a_i - D^i\xi_0^{E(m/2)}, \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0, & D^{m-m}\xi_0{}^0, & \cdots & \cdots & \cdots \\
D^{m-m}\xi_0{}^0, & D^{m-(m-1)}\xi_0, & \cdots D^{m-(j+1)}\xi_0^{E[(m-(j+1))/2]}, & a_m - D^m\xi_0^{E(m/2)},
\end{array}\right.$$

$$\left.\begin{array}{cccc}
\cdots & \cdots & 0, & D^0\xi_0^{E[(m-1)/2]} \\
\cdots & \cdots & D^0\xi_0^{E[(m-2)/2]}, & D^1\xi_0^{E[(m-1)/2]} \\
\cdots & \cdots & \cdots & \cdots \\
D^{i-(j-1)}\xi_0^{E[(m-(j-1))/2]}, & \cdots & D^{i-2}\xi_0^{E[(m-2)/2]}, & D^{i-1}\xi_0^{E[(m-1)/2]} \\
\cdots & \cdots & \cdots & \cdots \\
D^{m-(j-1)}\xi_0^{E[(m-(j-1))/2]}, & \cdots & D^{m-2}\xi_0^{E[(m-2)/2]}, & D^{m-1}\xi_0^{E[(m-1)/2]}
\end{array}\right| \div \Delta,$$

where $\Delta = \xi_0{}^m$.

If we expand the numerator of this expression in terms of the column $a_i - D^i\xi_0^{E(m/2)}$, the minors of the constituents $a_i - D^i\xi_0^{E(m/2)}$ are determinants of the type $(A)$ of § 1. So in the symbolic notation,

$$(2) \quad \alpha_{m-j+1} =$$

$$\frac{(-1)^{i+j}\sum_{i=1}^{j}\{(D^1(D^2\cdots(D^{i-1}(D^{i+1}\cdots(D^{m-1}(D^m)^m\xi_0{}^0)^{m-1}\xi_0{}^0 \cdots)^{j+1}\xi_0^{E[(m-(j+1))/2]})^{j-1}\xi_0^{E[(m-(j-1))/2]} \cdots)^2\xi_0^{E[(m-1)/2]})^1\xi_0^{E[(m-1)/2]}\}(a_i - D^i\xi_0^{E(m/2)})}{\xi_0{}^m}.$$

This result enables us to make a rapid computation of the $\alpha_i$ in particular cases.

Example 1. $m = 4, j = 3$.

$$\alpha_2 = [\{(D^2(D^3(D^4)^4\xi_0{}^0)^2\xi_0)^1\xi_0\}(a_1 - D\xi_0{}^2)$$
$$- \{(D^1(D^3(D^4)^4\xi_0{}^0)^2\xi_0)^1\xi_0\}(a_2 - D^2\xi_0{}^2)$$
$$+ \{(D^1(D^2(D^4)^4\xi_0{}^0)^2\xi_0)^1\xi_0\}(a_3 - D^3\xi_0{}^2)] \div \xi_0{}^4.$$
$$= [\{D^0\xi_0{}^0(D^2(D^3)^2\xi_0)^1\xi_0\}(a_1 - D^1\xi_0{}^2)$$

$$
\begin{aligned}
&\quad - D^0\xi_0{}^0 D^0\xi_0 (D^3)^2\xi_0 (a_2 - D^2\xi_0{}^2) \\
&\quad + D^0\xi_0{}^0 D^0\xi_0 D^0\xi_0 (a_3 - D^3\xi_0{}^2)]^* \div \xi_0{}^4 \\
&= [1(D\xi_0 \cdot D\xi_0 - D^0\xi_0 D^2\xi_0)(a_1 - D\xi_0{}^2) \\
&\quad - 1\cdot\xi_0\cdot D\xi_0 (a_2 - D^2\xi_0{}^2) + 1\cdot\xi_0\xi_0 (a_3 - D^3\xi_0{}^2)] \div \xi_0{}^4 \\
&= [(\xi_1{}^2 - \xi_0\xi_2)(a_1 - 2\xi_0\xi_1) - \xi_0\xi_1 (a_2 - \xi_1{}^2 - \xi_0\xi_2) \\
&\quad + \xi_0{}^2 (a_3 - \xi_1\xi_2)] \div \xi_0{}^4 \\
&= [(-\xi_0\xi_1{}^3 + a_1\xi_1{}^2 - \xi_0\xi_1 a_2 + \xi_0{}^2 a_3) + 2(\xi_0{}^2\xi_1 - \xi_0 a_1)\xi_2] \div \xi_0{}^4.
\end{aligned}
$$

Example 2. $m = 3, j = 3.$

$$
\begin{aligned}
\alpha_1 &= [(D^2(D^3)^2\xi_0{}^0)^1\xi_0 (a_1 - D\xi_0) - (D^1(D^3)^2\xi_0{}^0)^1\xi_0)(a_2 - D^2\xi_0) \\
&\qquad + (D^1(D^2)^2\xi_0{}^0)^1\xi_0 (a_3 - D^3\xi_0)] \div \xi_0{}^3 \\
&= [(D\xi_0 \cdot D\xi_0{}^0 - D^0\xi_0{}^0 D^2\xi_0)(a_1 - D\xi_0) \\
&\quad - D^0\xi_0 \cdot D\xi_0{}^0 (a_2 - D^2\xi_0) + D^0\xi_0 \cdot D^0\xi_0{}^0 (a_3 - D^3\xi_0)] \div \xi_0{}^3 \\
&= [(\xi_1 \cdot 0 - 1\cdot\xi_2)(a_1 - \xi_1) - \xi_0 \cdot 0(a_2 - \xi_2) \\
&\qquad + \xi_0 \cdot 1(a_3 - 0)] \div \xi_0{}^3 \\
&= [\xi_0 a_3 - (-\xi_1 + a_1)\xi_2] \div \xi_0{}^3.
\end{aligned}
$$

Without interfering with the generality of the results we may take $\xi_0 = 1$, and then the values for $\alpha_1$ and $\alpha_2$ can be put in the following form:†

$$
(3) \qquad
\begin{aligned}
\alpha_1 &= a_m - \sum_{i=0}^{E[(m-1)/2]} \frac{\xi_2{}^{i+1}}{\lfloor i} \Omega_m{}^{i+2} a_{-\xi_1}{}^{m(i)}, \ddagger \\
\alpha_2 &= \sum_{i=0}^{E[(m-1)/2]} \frac{\xi_2{}^{i}}{\lfloor i} \Omega_m{}^{i+1} a_{-\xi_1}{}^{m(i)},
\end{aligned}
$$

where

$$
\Omega_m = a_0 \frac{\partial}{\partial a_1} + a_1 \frac{\partial}{\partial a_2} + \cdots + a_{m-1} \frac{\partial}{a_m}.
$$

Thus, for example, the results in the particular case, $m = 4$,

* $(D^i)^i\xi_0{}^j = D^{i-i}\xi_0{}^j = D^0\xi_0{}^j$ is factored out of each term since it enters the result as a constant multiplier.

† Cf. *Transactions Amer. Math. Soc.*, Vol. 15 (1914), No. 1, p. 79.

‡ $-\xi_1$ is substituted for $x$ before the differentiation is performed and $(i)$ is the power of the differential coefficient.

can be written

$$\alpha_1 = a_4 - \sum_{i=0}^{1} \frac{\xi_2^{i+1}}{\lfloor i} \Omega_4^{i+2} a_{-\xi_1}^{4(i)},$$

$$\alpha_2 = \sum_{i=0}^{1} \frac{\xi_2^{i}}{\lfloor i} \Omega_4^{i+1} a_{-\xi_1}^{4(i)}, \tag{4}$$

or

$$\begin{aligned} \alpha_1 &= a_4 - \xi_2 \Omega_4^2 a_{-\xi_1}^4 - \xi_2^2 \Omega_4^3 a_{-\xi_1}^{4(1)} \\ &= a_4 - (a_0\xi_1^2 - a_1\xi_1 + a_2)\xi_2 + a_0\xi_2^2, \\ \alpha_2 &= \Omega_4 a_{-\xi_1}^4 + \xi_2 \Omega_4^2 a_{-\xi_1}^{4(1)} \\ &= (-a_0\xi_1^3 + a_1\xi_1^2 - a_2\xi_1 + a_3) + (2a_0\xi_1 - a_1)\xi_2. \end{aligned} \tag{4a}$$

This result agrees with that of example 1 if we make $\xi_0 = 1$. From (3)

$$F = (\alpha_1 + \alpha_2 x) = a_m + \sum_{i=0}^{E[(m-1)/2]} \frac{-\xi_2^{i+1}}{\lfloor i} \Omega_m^{i+2} a_{-\xi_1}^{m(i)} + \frac{\xi_2^{i}}{\lfloor i} \Omega_m^{i+1} a_{-\xi_1}^{m(i)} x. \tag{5}$$

There is proved in the paper just cited a general result* which is applied here for the case $\nu = 2$; namely,

$$(\alpha_{2k+1} + \alpha_{2k+2}x) = -\frac{1}{k}\frac{\partial}{\partial \xi_2}(\alpha_{2k-1} + \alpha_{2k}x).$$

Hence the assumed expansions (1) and (1a) take the form

$$\begin{aligned} a_x^m = F &- \frac{\partial F}{\partial \xi_2}(\xi_x^2) + \frac{1}{\lfloor 2}\frac{\partial^2 F}{\partial \xi_2^2}(\xi_x^2)^2 + \\ &\cdots + \frac{(-1)^k}{\lfloor k}\frac{\partial^k F}{\partial^k \xi_2}(\xi_x^2)^k + \cdots - \frac{(-1)^{E(m/2)}}{\lfloor E(m/2)}\frac{\partial^{E(m/2)}}{\partial \xi_2^{E(m/2)}}(\xi_x^2)^{E(m/2)}. \end{aligned} \tag{6}$$

The coefficients in this expansion are symmetric functions of the roots of $\xi_x^2$ and also of those of $a_x^m$. Expressed in terms of the coefficients of $a_x^m$, and $\xi_x^2$, they are linear in $a_0, a_1, \cdots, a_m$, but, in general, not linear in $\xi_0, \xi_1, \xi_2$. These functions have a seminvariant character as appears in the application which follows.

* In Glenn's paper the corresponding expansions of the general form of order $\mu\nu$ in the general argument $\xi_x^\nu$ is given.

## § 3. CONDITION THAT A FORM CONTAIN A GIVEN QUADRATIC FORM AS A FACTOR.

Using the expansion (6) obtained in § 2, a minimum set of necessary and sufficient conditions that a given form be divisible by the $r$th power of a quadratic form is the identical vanishing of $F$, $\frac{\partial F}{\partial \xi_2}$, $\frac{\partial^2 F}{\partial \xi_2^2}$, $\cdots$ $\frac{\partial^{r-1} F}{\partial \xi_2^{r-1}}$. Since these expressions are linear in $x$ and do not vanish identically for every value of $x$, then the identical vanishing of each of these $r$ expressions necessitates the vanishing of two seminvariants, and hence the simultaneous vanishing of $2r$ seminvariants furnishes the minimum set of necessary and sufficient conditions that $a_x^m$ be divisible by the $r$th power of $\xi_x^2$. In particular, if $a_x^m$ contain $\xi_x^2$ as a factor, then from (4) and (5)

$$(7)\qquad \begin{aligned} \alpha_1 &= a_m - \sum_{i=0}^{E(m-1/2)} \frac{\xi_2^{i+1}}{\lfloor i} \Omega_m{}^{i+2} a_{-\zeta_1}^{m(i)} = 0, \\ \alpha_2 &= \sum_{i=0}^{E(m-1/2)} \frac{\xi_2^{i}}{\lfloor i} \Omega_m{}^{i+1} a_{-\xi_1}^{m(i)} = 0. \end{aligned}$$

and conversely. These two conditions are equivalent to the redundant set of $m+1$ conditions furnished by the identical vanishing of the covariant $\Phi$, given by Clebsch,* which for $m = 4$ is

$$\Phi = -2D(a\xi)\xi_x \cdot a_x^3 + 2\xi_x^2(a\xi')(a\xi'')\xi_x' a_x = 0,$$

where $D$ is the discriminant of $\xi_x^2$ and $\xi_x^2 = \xi_x'^2 = \xi_x''^2$. B. Igel† has developed a method depending upon the vanishing of a series of resultants of every two forms of a certain system.

* Theorie der Binaren Formen, p. 94.

† *Sitzungsberichte der Kön. Akad. der Wissenshaften Mathematische.*, Bd. LXXXII, p. 943.

## § 4. SEPARATION OF $a_x^m/(\xi_x^2)^k$ INTO PARTIAL FRACTIONS.

From (6) § 2, we have

$$\frac{a_x^m}{(\xi_x^2)^k} = \frac{F - \dfrac{\partial F}{\partial \xi_2}\xi_x^2 + \dfrac{1}{\lfloor 2}\dfrac{\partial^2 F}{\partial \xi_2^2}(\xi_x^2)^2 + \cdots + \dfrac{(-1)^k}{\lfloor k}\dfrac{\partial^k F}{\partial \xi_2^k}(\xi_x^2)^k + \cdots \dfrac{(-1)^n}{\lfloor n}\dfrac{\partial^n F}{\partial \xi_2^n}(\xi_x^2)^n}{(\xi_x^2)^k}.$$

Hence

$$\frac{a_x^m}{(\xi_x^2)^k} = \frac{F}{(\xi_x^2)^k} - \frac{\dfrac{\partial F}{\partial \xi_2}}{(\xi_x^2)^{k-1}} + \cdots \frac{(-1)^{k-1}}{\lfloor k-1}\frac{\dfrac{\partial^{k-1}F}{\partial \xi_2^{k-1}}}{\xi_x^2} + \frac{(-1)^k}{\lfloor k}\cdot\frac{\partial^k F}{\partial^k \xi_2} + \frac{(-1)^{k+1}}{\lfloor k+1}\frac{\partial^{k+1}F}{\partial \xi_2^{k+1}}\cdot(\xi_x^2) + \cdots \frac{(-1)^n}{\lfloor n}\frac{\partial^n F}{\partial \xi_2^n}\cdot(\xi_x^2)^{n-k},$$

or, upon writing the numerator in explicit form,

$$\frac{a_x^m}{(\xi_x^2)^k} = \frac{a_m + \sum_{i=0}^{E[(m-1)/2]} -\dfrac{{\xi_2}^{i+1}}{\lfloor i}\Omega_m{}^{i+2}a_{-\xi_1}^{m(i)} + \dfrac{{\xi_2}^i}{\lfloor i}\Omega_m{}^{i+1}a_{-\xi_1}^{m(i)}x}{(\xi_x^2)^k} + \sum_{s=1}^{E(m/2)}\frac{\sum_{i=0}^{E[(m-1)/2]}\dfrac{(i+1){\xi_2}^{i-s+1}}{\lfloor i-s}\Omega_m{}^{i+2}a_{-\xi_1}^{m(i)} + \dfrac{{\xi_2}^{i-s}}{\lfloor i-s}\Omega_m{}^{i+1}a_{-\xi_1}^{m(i)}}{(\xi_x^2)^{k-s}}.$$

To illustrate, let

$$\frac{a_x^4}{(\xi_x^2)^3} = \frac{x^4 - x^3 + 3x^2 - x}{(x^2 - x + 1)^3} = \frac{F}{(x^2 - x + 1)^3} - \frac{\partial F/\partial \xi_2}{(x^2 - x + 1)^2} + \frac{\partial^2 F/\partial \xi_2^2}{\lfloor 2(x^2 - x + 1)}.$$

From (5)

$$F = -(\xi_1^2 + \xi_1 + 3)\xi_2 + \xi_2^2 + \{(-\xi_1^3 - \xi_1^2 - 3\xi_1 - 1) + (2\xi_1 + 1)\xi_2\}x,$$

$$\frac{\partial F}{\partial \xi_2} = -(\xi_1^2 + \xi_1 + 3) + 2\xi_2 + (2\xi_1 + 1)x,$$

$$\frac{\partial^2 F}{\lfloor 2 \partial \xi_2^2} = 1.$$

Now, $\xi_1 = -1$, $\xi_2 = 1$.

$$\therefore \quad F = x - 2, \quad \frac{\partial F}{\partial \xi_2} = -x - 1, \quad \frac{1}{\lfloor 2} \frac{\partial^2 F}{\partial \xi_2^2} = 1.$$

Then

$$\frac{x^4 - x^3 + 3x^2 - x}{(x^2 - x + 1)^3} = \frac{x - 2}{(x^2 - x + 1)^3} + \frac{x + 1}{(x^2 - x + 1)^2} + \frac{1}{x^2 - x + 1}.$$

## § 5. COMPUTATION OF SYMMETRIC FUNCTIONS.

The two seminvariants obtained in § 3, whose identical vanishing furnished the necessary and sufficient conditions that a form of order $m$ be divisible by a given quadratic form, also furnish a direct method for the computation of certain classes of symmetric functions.

Let the $m$ roots of

$$f(x) = a_x^m = a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \cdots + a_m = 0$$

be $x_1, x_2, \cdots x_m$, and denote the totality of the sums of these roots taken in pairs by $\xi_{1i}$ and the products in pairs by $\xi_{2j}$. $[i, j = 1, 2, \cdots \frac{1}{2}m(m-1).]$

The number of quadratic forms which will divide $a_x^m$ is $\frac{1}{2}m(m-1)$ and they are of the form

$$x^2 - (x_i + x_j)x + x_i x_j = x^2 + \xi_{1k} x + \xi_{2k}. \tag{8}$$

Hence, if from the two conditions (7),

$$\alpha_1 = a_m - \sum_{i=0}^{E[(m-1)/2]} \frac{\xi_2^{i+1}}{\lfloor i} \Omega_m^{i+2} a_{-\xi_1}^{m(i)} = 0,$$

$$\alpha_2 = \sum_{i=0}^{E[(m-1)/2]} \frac{\xi_2^{i}}{\lfloor i} \Omega_m^{i+1} a_{-\xi_1}^{m(i)} = 0,$$

where $\alpha_1$ is of order $m-2$ in $\xi_1$ and $E\left(\frac{m+1}{2}\right)$ in $\xi_2$, and $\alpha_2$ is of order $m-1$ in $\xi_1$ and $E\left(\frac{m-1}{2}\right)$ in $\xi_2$, we eliminate $\xi_2$,

we obtain a resolvent of order $\frac{1}{2}m(m-1)$,

$$\Phi = \sum_{i=0}^{\frac{1}{2}m(m-1)} (-1)^i \phi_i \xi_1^{\frac{1}{2}m(m-1)-i} = 0, \tag{9}$$

whose roots are the $\xi_{1i}$. The $\phi_i$ are values of the elementary symmetric functions of the $\xi_{1i}$ and are rational functions of the $a$'s. By eliminating $\xi_1$, we obtain in the same manner

$$\Psi = \sum_{j=0}^{\frac{1}{2}m(m-1)} (-1)^j \psi_j \xi_2^{\frac{1}{2}m(m-1)-j} = 0, \tag{10}$$

a resolvent whose roots are the $\xi_{2j}$. The $\psi_j$ are the values of the elementary symmetric functions of $\xi_{2j}$ and are expressible rationally in terms of the $a$'s.

## § 6. THE RESOLVENTS OF KÖNIG.

The $\Psi = 0$ of § 5, is the König resolvent for the case $\nu = 2$. $\Phi = 0$ is a resolvent of a similar nature. Both of these resolvents have been studied by Glenn* in his theory of the rational resolvents of the factorable ternary forms.

The general form of the $\Phi$ resolvent is, from (7)

$$\Phi\dagger = \begin{vmatrix} \Omega a^m_{-\xi_1}, & \Omega^2 a^{m(1)}_{-\xi_1}, & \cdots & \Omega^{E(m+1/2)} a^{m[E(m-1)/2]}_{-\xi_1}, & 0, & \cdots & 0 \\ 0, & \Omega a^m_{-\xi_1}, & \Omega^2 a^{m(1)}_{-\xi_1}, & \cdots & \Omega^{E(m+1/2)} a^{m[E(m-1/2)]}_{-\xi_1}, & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_m, & \Omega^2 a^m_{-\xi_1}, & \Omega^3 a^{m(1)}_{-\xi_1}, & \cdots & \Omega^{E(m+3/2)} a^{m\lfloor E(m-1/2)]}_{-\xi_1}, & \cdots, & 0 \\ 0, & a_m, & \Omega^2 a^m_{-\xi_1}, & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0, & 0, & \cdots & \cdots & \Omega^{E(m+3/2)} a^{m[E(m-1/2)]}_{-\xi_1} & & \end{vmatrix} = 0.$$

The elimination of $\xi_1$ from equations (7) does not exhibit the $\Psi\ddagger$ resolvent in a concise form like the determinant above.

The determination of the $\frac{1}{2}m(m-1)$ quadratic forms which divide a given form can be obtained by solving either $\Phi = 0$ or $\Psi = 0$. If $\Phi = 0$ be solved, then by substituting the $\xi_{1i}$ in the condition $\alpha_2$ of (3), $\xi_{2i}$ can be obtained and the quadratic factor is determined.

* *Amer. Jour. of Math.*, Vol. 32 (1910), p. 80.

† The order of the determinant is $m - 1$.

‡ The order of the $\Psi$ determinant is $m - 1 + m - 2 = 2m - 3$.

These resolvents are given for the case $m = 4$. From $(4^a)$,

$$\alpha_1 = -a_0\xi_2^2 + (a_0\xi_1^2 - a_1\xi_1 + a_2)\xi_2 + a_4 = 0,$$

$$\alpha_2 = (2a_0\xi_1 - a_1)\xi_2 + (-a_0\xi_1^3 + a_1\xi_1^2 - a_2\xi_1 + a_3) = 0.$$

Eliminating $\xi_2$,

$$\Phi = \begin{vmatrix} -a_0, & a_0\xi_1^2 - a_1\xi_1 + a_2, & a_4 \\ 2a_0\xi_1 - a_1, & -a_0\xi_1^3 + a_1\xi_1^2 - a_2\xi_1 + a_3, & 0 \\ 0, & 2a_0\xi_1 - a_1, & -a_0\xi_1^3 + a_1\xi_1^2 - a_2\xi_1 + a_3 \end{vmatrix} = 0.$$

Likewise, eliminating $\xi_1$,

$$\Psi = \begin{vmatrix} a_0\xi_2, & -a_1\xi_2, & -a_0\xi_2^2 + a_2\xi_2 + a_4, & 0 & 0 \\ 0, & a_0\xi_2, & -a_1\xi_2, & -a_0\xi_2^2 + a_2\xi_2 + a^4, & 0 \\ 0, & 0, & a_0\xi_2, & -a_1\xi_2, & -a_0\xi_2^2 + a_2\xi_2 + a_4 \\ -a_0, & a_1, & 2a_0\xi_2 - a_2, & -a_1\xi_2 + a_3, & 0 \\ 0, & -a_0, & a_1, & 2a_0\xi_2 - a_2, & -a_1\xi_2 + a_3 \end{vmatrix} = 0.$$

The explicit expanded forms of $\Phi = 0$ and $\Psi = 0$ can be obtained from the tables in § 7.

## § 7. TABLES OF SYMMETRIC FUNCTIONS OF THE SUMS AND OF THE PRODUCTS OF THE ROOTS OF A GIVEN FORM, TAKEN IN PAIRS.

In § 5, it was shown that there existed two resolvents $\Phi = 0$ and $\Psi = 0$, whose roots were the $\xi_{1i}$ and the $\xi_{2j}$ of (8) respectively, and hence the coefficients of the powers of $\xi_1$ in $\Phi = 0$ and $\xi_2$ in $\Psi = 0$ are the values of the elementary symmetric functions of $\xi_{1i}$ and $\xi_{2j}$, respectively, expressed rationally in terms of the $a$'s of $f = a_x^m$. These symmetric functions have been computed

for the cases $m = 3, 4, 5, 6$ and a table of them in homogeneous form eompletes this paper. In order to eheck these results, the following formulas have been derived:

$$\phi_k = \sum_i \lambda_i {a_0}^{p_i} {a_1}^{q_i} \cdots {a_m}^{t_i},$$

(a)
$$(-2)^k \binom{{}_mC_2}{k} = \sum_i \lambda_i \binom{m}{0}^{p_i} \binom{m}{1}^{q_i} \cdots \binom{m}{m}^{t_i}.$$

$$\psi_k = \sum_i \mu_i {a_0}^{p_i} {a_1}^{q_i} \cdots {a_m}^{t_i},$$

(b)
$$\binom{{}_mC_2}{k} = \sum_i \mu_i \binom{m}{0}^{p_i} \binom{m}{1}^{q_i} \cdots \binom{m}{m}^{t_i}.$$

As a further eheck upon the $\psi$'s, it may be notieed that by interehanging $a_0$ with $a_m$, $a_1$ with $a_{m-1}$, ete., $\psi_j$ is transformed into $\psi_{m-j}$. $(j = 0, 1, 2, \cdots m.)$

Table Ia.* $m = 3$.

$$\phi_1 = 2a_0a_1,$$
$$\phi_2 = {a_1}^2 + a_0a_2,$$
$$\phi_3 = a_1a_2 - a_0a_3.$$

Table Ib.

$$\psi_1 = a_0a_2,$$
$$\psi_2 = a_1a_3,$$
$$\psi_3 = {a_3}^2.$$

Table IIa. $m = 4$.

| $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ | $\phi_5$ | $\phi_6$ |
|---|---|---|---|---|---|
| $3{a_0}^2a_1$ | $2{a_0}^2a_2$<br>$+ 3a_0{a_1}^2$ | $4a_0a_1a_2$<br>$+ {a_1}^3$ | $- 4{a_0}^2a_4$<br>$+ a_0a_1a_3$<br>$+ a_0{a_2}^2$<br>$+ 2{a_1}^2a_2$ | $- 4a_0a_1a_4$<br>$+ a_1{a_2}^2$ | $- a_0{a_3}^2$<br>$- {a_1}^2a_4$<br>$+ a_1a_2a_3$ |

Table IIb.

| $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ | $\psi_5$ | $\psi_6$ |
|---|---|---|---|---|---|
| ${a_0}^2a_2$ | $- {a_0}^2a_4$<br>$+ a_0a_1a_3$ | $- 2a_0a_2a_4$<br>$+ a_0{a_3}^2$<br>$+ {a_1}^2a_4$ | $- a_0{a_4}^2$<br>$+ a_1a_3a_4$ | $+ a_2{a_4}^2$ | ${a_4}^3$ |

* $\phi_0 = \psi_0 = {a_0}^{m-1}$.

TABLE IIIa. $m = 5$.

| $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ | $\phi_5$ |
|---|---|---|---|---|
| $4a_0^3a_1$ | $3a_0^3a_2$<br>$+ 6a_0^2a_1^2$ | $a_0^3a_3$<br>$+ 9a_0^2a_1a_2$<br>$+ 4a_0a_1^3$ | $- 3a_0^3a_4$<br>$+ 4a_0^2a_1a_3$<br>$+ 3a_0^2a_2^2$<br>$+ 9a_0a_1^2a_2$<br>$+ a_1^4$ | $- 11a_0^3a_5$<br>$- 5a_0^2a_1a_4$<br>$+ 2a_0^2a_2a_3$<br>$+ 5a_0a_1^2a_3$<br>$+ 6a_0a_1a_2^2$<br>$+ 3a_1^3a_2$ |

TABLE IIIa (*continued*).

| $\phi_6$ | $\phi_7$ | $\phi_8$ | $\phi_9$ | $\phi_{10}$ |
|---|---|---|---|---|
| $- 22a_0^2a_1a_5$<br>$- 2a_0^2a_2a_4$<br>$- a_0^2a_3^2$<br>$- 2a_0a_1^2a_4$<br>$+ 6a_0a_1a_2a_3$<br>$+ a_0a_2^3$<br>$+ 2a_1^3a_3$<br>$+ 3a_1^2a_2^2$ | $- 4a_0^2a_2a_5$<br>$- 4a_0^2a_3a_4$<br>$- 16a_0a_1^2a_5$<br>$+ a_0a_2^2a_3$<br>$+ 4a_1^2a_2a_3$<br>$+ a_1a_2^3$ | $7a_0^2a_3a_5$<br>$- 4a_0^2a_4^2$<br>$- 9a_0a_1a_2a_5$<br>$- 3a_0a_1a_3a_4$<br>$+ a_0a_2^2a_4$<br>$- a_0a_2a_3^2$<br>$- 4a_1^3a_5$<br>$+ a_1^2a_2a_4$<br>$+ a_1^2a_3^2$<br>$+ 2a_1a_2^2a_3$ | $4a_0^2a_4a_5$<br>$+ 4a_0a_1a_3a_5$<br>$- 4a_0a_1a_4^2$<br>$- a_0a_2^2a_5$<br>$- a_0a_3^3$<br>$- 4a_1^2a_2a_5$<br>$- a_1a_2^2a_4$<br>$+ a_1a_2a_3^2$ | $- a_0^2a_5^2$<br>$+ 2a_0a_1a_4a_5$<br>$+ a_0a_2a_3a_5$<br>$- a_0a_3^2a_4$<br>$- a_1^2a_4^2$<br>$- a_1a_2^2a_5$<br>$+ a_1a_2a_3a_4$ |

TABLE IIIb.

| $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ | $\psi_5$ |
|---|---|---|---|---|
| $a_0^3a_2$ | $- a_0^3a_4$<br>$+ a_0^2a_1a_3$ | $- a_0^2a_1a_5$<br>$- 2a_0^2a_2a_4$<br>$+ a_0^2a_3^2$<br>$+ a_0a_1^2a_4$ | $+ a_0^2a_3a_5$<br>$- a_0^2a_4^2$<br>$- 3a_0a_1a_2a_5$<br>$+ a_0a_1a_3a_4$<br>$+ a_1^3a_5$ | $2a_0^2a_5^2$<br>$- 2a_0a_1a_4a_5$<br>$- 2a_0a_2a_3a_5$<br>$+ a_0a_2a_4^2$<br>$+ a_1^2a_3a_5$ |

TABLE IIIb (*continued*).

| $\psi_6$ | $\psi_7$ | $\psi_8$ | $\psi_9$ | $\psi_{10}$ |
|---|---|---|---|---|
| $a_0a_2a_5^2$<br>$- 3a_0a_3a_4a_5$<br>$+ a_0a_4^3$<br>$- a_1^2a_5^2$<br>$+ a_1a_2a_4a_5$ | $- a_0a_4a_5^2$<br>$- 2a_1a_3a_5^2$<br>$+ a_1a_4^2a_5$<br>$+ a_2^2a_5^2$ | $- a_1a_5^3$<br>$+ a_2a_4a_5^2$ | $a_3a_5^3$ | $a_5^4$ |

TABLE IVa. $m = 6$.

| $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ | $\phi_5$ | $\phi_6$ | $\phi_7$ | $\phi_8$ |
|---|---|---|---|---|---|---|---|
| $5a_0^4a_1$ | $4a_0^4a_2$<br>$+ 10a_0^3a_1^2$ | $2a_0^4a_3$<br>$+ 16a_0^3a_1a_2$<br>$+ 10a_0^2a_1^3$ | $- 2a_0^4a_4$<br>$+ 9a_0^3a_1a_3$<br>$+ 6a_0^3a_2^2$<br>$+ 24a_0^2a_1^2a_2$<br>$+ 5a_0a_1^4$ | $- 10a_0^4a_5$<br>$- 4a_0^3a_1a_4$<br>$+ 6a_0^3a_2a_3$<br>$+ 15a_0^2a_1^2a_3$<br>$+ 18a_0^2a_1a_2^2$<br>$+ 16a_0a_1^3a_2$<br>$+ a_1^5$ | $- 26a_0^4a_6$<br>$- 29a_0^3a_1a_5$<br>$- 2a_0^3a_2a_4$<br>$- a_0^2a_1^2a_4$<br>$+ 21a_0^2a_1a_2a_3$<br>$+ 4a_0^2a_2^3$<br>$+ 11a_0a_1^3a_3$<br>$+ 18a_0a_1^2a_2^2$<br>$+ 4a_1^4a_2$ | $- 78a_0^3a_1a_6$<br>$- 12a_0^3a_2a_5$<br>$- 6a_0^3a_3a_4$<br>$- 32a_0^2a_1^2a_5$<br>$+ 3a_0^2a_1a_3^2$<br>$+ 2a_0^2a_1a_2a_4$<br>$+ 6a_0^2a_2^2a_3$<br>$+ 2a_0a_1^3a_4$<br>$+ 24a_0a_1^2a_2a_3$<br>$+ 8a_0a_1a_2^3$<br>$+ 3a_1^4a_3$<br>$+ 6a_1^3a_2^2$ | $- 24a_0^3a_2a_6$<br>$- 3a_0^3a_3a_5$<br>$- 7a_0^3a_4^2$<br>$- 94a_0^2a_1^2a_6$<br>$- 26a_0^2a_1a_2a_5$<br>$- 8a_0^2a_1a_3a_4$<br>$+ 2a_0^2a_2^2a_4$<br>$- 16a_0a_1^3a_5$<br>$+ 8a_0a_1^2a_2a_4$<br>$+ 6a_0a_1^2a_3^2$<br>$+ 15a_0a_1a_2^2a_3$<br>$+ a_0a_4^2$<br>$+ a_1^4a_4$<br>$+ 9a_1^3a_2a_3$<br>$+ 4a_1^2a_2^3$ |

TABLE IVa (*continued*).

| $\phi_9$ | $\phi_{10}$ | $\phi_{11}$ | $\phi_{12}$ | $\phi_{13}$ | $\phi_{14}$ | $\phi_{15}$ |
|---|---|---|---|---|---|---|
| $3a_0^3a_3a_6$ | $54a_0^3a_4a_6$ | $18a_0^3a_5a_6$ | $-27a_0^3a_6^2$ | $-27a_0^2a_1a_6^2$ | $-9a_0a_3a_5a_6$ | $-a_0^2a_5^3$ |
| $-10a_0^3a_4a_5$ | $-12a_0^3a_5^2$ | $+84a_0^2a_1a_4a_6$ | $+33a_0^2a_1a_5a_6$ | $+6a_0^2a_3a_5^2$ | $+3a_0^2a_4a_5^2$ | $+3a_0a_1a_3a_5a_6$ |
| $-76a_0^2a_1a_2a_6$ | $+33a_0^2a_1a_3a_6$ | $-23a_0^2a_1a_5^2$ | $+18a_0^2a_2a_4a_6$ | $+21a_0a_1^2a_5a_6$ | $-9a_0a_1^2a_6^2$ | $+2a_0a_1a_4a_5^2$ |
| $-7a_0^2a_1a_3a_5$ | $-21a_0^2a_1a_4a_5$ | $+6a_0^2a_2a_3a_6$ | $-5a_0^2a_2a_5^2$ | $+18a_0a_1a_2a_4a_6$ | $+6a_0a_1a_2a_5a_6$ | $+a_0a_2a_3a_5^2$ |
| $-13a_0^2a_1a_4^2$ | $-18a_0^2a_2^2a_6$ | $-2a_0^2a_2a_4a_5$ | $+3a_0^2a_3^2a_6$ | $-9a_0a_1a_2a_5^2$ | $+3a_0a_1a_3a_4a_6$ | $-a_0a_3^3a_6$ |
| $-2a_0^2a_2^2a_5$ | $+6a_0^2a_2a_3a_5$ | $+6a_0^2a_3^2a_5$ | $+7a_0^2a_3a_4a_5$ | $+3a_0a_1a_3^2a_6$ | $+4a_0a_1a_3a_5^2$ | $-a_0a_3^2a_4a_5$ |
| $-4a_0^2a_2a_3a_4$ | $-6a_0^2a_2a_4^2$ | $-8a_0^2a_3a_4^2$ | $-4a_0^2a_4^3$ | $+3a_0a_1a_3a_4a_5$ | $-2a_0a_1a_4^2a_5$ | $-a_1^3a_6^2$ |
| $-2a_0^2a_3^3$ | $-6a_0^2a_3^2a_4$ | $+4a_0a_1^2a_3a_6$ | $+43a_0a_1^2a_4a_6$ | $-4a_0a_1a_4^3$ | $-a_0a_2^2a_5^2$ | $-2a_1^2a_2a_5a_6$ |
| $-52a_0a_1^3a_6$ | $-72a_0a_1^2a_2a_6$ | $-16a_0a_1^2a_4a_5$ | $-16a_0a_1^2a_5^2$ | $-2a_0a_3^3a_4$ | $+3a_0a_2a_3^2a_6$ | $-a_1^2a_3a_4a_6$ |
| $-20a_0a_1^2a_2a_5$ | $-7a_0a_1^2a_3a_5$ | $-34a_0a_1a_2^2a_6$ | $-5a_0a_1a_2a_3a_6$ | $+7a_1^3a_4a_6$ | $+a_0a_2a_3a_4a_5$ | $-a_1^2a_4^2a_5$ |
| $+8a_0a_1a_2^2a_4$ | $-7a_0a_1^2a_4^2$ | $+2a_0a_1a_2a_3a_5$ | $-7a_0a_1a_2a_4a_5$ | $-4a_1^3a_5^2$ | $-a_0a_3^3a_5$ | $-a_1a_2^2a_5^2$ |
| $-6a_0a_1a_2a_3^2$ | $-5a_0a_1a_2^2a_5$ | $-4a_0a_1a_2a_4^2$ | $+4a_0a_1a_3^2a_5$ | $-9a_1^2a_2a_3a_6$ | $-a_0a_3^2a_4^2$ | $+a_1a_2a_3^2a_6$ |
| $+2a_0a_2^3a_3$ | $+4a_0a_1a_2a_3a_4$ | $-4a_0a_1a_3^2a_4$ | $-7a_0a_1a_3a_4^2$ | $-3a_1^2a_2a_4a_5$ | $+4a_1^3a_5a_6$ | $+a_1a_2a_3a_4a_5$ |
| $-3a_1^4a_5$ | $-a_0a_1a_3^3$ | $+2a_0a_2^2a_3a_4$ | $-4a_0a_2^3a_6$ | $+a_1^2a_3^2a_5$ | $+4a_1^2a_2a_4a_6$ | |
| $+4a_1^3a_2a_4$ | $+2a_0a_2^3a_4$ | $-2a_0a_2a_3^3$ | $+a_0a_2^2a_3a_5$ | $-a_1^2a_3a_4^2$ | $-4a_1^2a_2a_5^2$ | |
| $+3a_1^3a_3^2$ | $-11a_1^4a_6$ | $-22a_1^3a_2a_6$ | $+a_0a_2^2a_4^2$ | $-4a_1a_2^3a_6$ | $-a_1^2a_3^2a_6$ | |
| $+9a_1^2a_2^2a_3$ | $-5a_1^3a_2a_5$ | $-2a_1^3a_3a_5$ | $-2a_0a_2a_3^2a_4$ | $+a_1a_2^2a_3a_5$ | $-a_1^2a_4^3$ | |
| $+a_1a_2^4$ | $+2a_1^3a_3a_4$ | $-a_1^3a_4^2$ | $-a_0a_3^4$ | $+a_1a_2^2a_4^2$ | $-4a_1a_2^2a_3a_6$ | |
| | $+5a_1^2a_2^2a_4$ | $-2a_1^2a_2^2a_5$ | $-4a_1^3a_3a_6$ | $+2a_1a_2a_3^2a_4$ | $+a_1a_2a_3^2a_5$ | |
| | $+6a_1^2a_2a_3^2$ | $+6a_1^2a_2a_3a_4$ | $-4a_1^3a_4a_5$ | | $+a_1a_2a_3a_4^2$ | |
| | $+3a_1a_2^3a_3$ | $+a_1^2a_3^3$ | $-16a_1^2a_2^2a_6$ | | | |
| | | $+2a_1a_2^3a_4$ | $+a_1^2a_3^2a_4$ | | | |
| | | $+3a_1a_2^2a_3^2$ | $+4a_1a_2^2a_3a_4$ | | | |
| | | | $+a_1a_2a_3^3$ | | | |

TABLE IVb.

| $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ | $\psi_5$ | $\psi_6$ | $\psi_7$ | $\psi_8$ |
|---|---|---|---|---|---|---|---|
| $a_0^4a_2$ | $-a_0^4a_4$ | $a_0^4a_6$ | $2a_0^3a_2a_6$ | $-2a_0^3a_4a_6$ | $-2a_0^2a_6^2$ | $+3a_0^2a_3a_5a_6$ | $+a_0^2a_5^2a_6$ |
| | $+a_0^3a_1a_3$ | $-2a_0^3a_1a_5$ | $+a_0^3a_3a_5$ | $+2a_0^3a_5^2$ | $+3a_0^2a_1a_5a_6$ | $+a_0^2a_4^2a_6$ | $+3a_0a_1a_3a_6^2$ |
| | | $-2a_0^3a_2a_4$ | $-a_0^3a_4^2$ | $+3a_0^2a_1a_3a_6$ | $+a_0^2a_2a_5^2$ | $-a_0^2a_4a_5^2$ | $+a_0a_1a_4a_5a_6$ |
| | | $+a_0^3a_3^2$ | $-a_0^2a_1^2a_6$ | $-2a_0^2a_1a_4a_5$ | $+3a_0^2a_3^2a_6$ | $+a_0a_1^2a_6^2$ | $-a_0a_1a_5^3$ |
| | | $+a_0^2a_1^2a_4$ | $-3a_0^2a_1a_2a_5$ | $+2a_0^2a_2^2a_6$ | $-3a_0^2a_3a_4a_5$ | $+a_0a_1a_2a_5a_6$ | $+a_0a_2^2a_6^2$ |
| | | | $+a_0^2a_1a_3a_4$ | $-2a_0^2a_2a_3a_5$ | $+a_0^2a_4^3$ | $-a_0a_1a_3a_4a_6$ | $-a_0a_2a_3a_5a_6$ |
| | | | $+a_0a_1^3a_5$ | $+a_0^2a_2a_4^2$ | $-a_0a_1^2a_4a_6$ | $-2a_0a_1a_3a_5^2$ | $-2a_0a_2a_4^2a_6$ |
| | | | | $-4a_0a_1^2a_2a_6$ | $-a_0a_1^2a_5^2$ | $+a_0a_1a_4^2a_5$ | $+a_0a_2a_4a_5^2$ |
| | | | | $+a_0a_1^2a_3a_5$ | $-3a_0a_1a_2a_5a_6$ | $-2a_0a_2^2a_4a_6$ | $-a_1^2a_2a_6^2$ |
| | | | | $+a_1^4a_6$ | $+a_0a_1a_2a_4a_5$ | $+a_0a_2^2a_5^2$ | $-2a_1^2a_3a_5a_6$ |
| | | | | | $+a_1^3a_3a_6$ | $-a_1^3a_5a_6$ | $+a_1^2a_4^2a_6$ |
| | | | | | | $+a_1^2a_2a_4a_6$ | $+a_1a_2^2a_5a_6$ |

TABLE IVb (*continued*).

| $\psi_9$ | $\psi_{10}$ | $\psi_{11}$ | $\psi_{12}$ | $\psi_{13}$ | $\psi_{14}$ | $\psi_{15}$ |
|---|---|---|---|---|---|---|
| $-2a_0^2a_6^3$ | $-2a_0a_2a_6^3$ | $+2a_0a_4a_6^3$ | $+a_0a_6^4$ | $-a_2a_6^4$ | $+a_4a_6^4$ | $a_6^5$ |
| $+3a_0a_1a_5a_6^2$ | $+3a_0a_3a_5a_6^2$ | $-a_0a_5^2a_6^2$ | $-a_1a_5a_6^3$ | $+a_3a_5a_6^3$ | | |
| $-a_0a_2a_5^2a_6$ | $+2a_0a_4^2a_6^2$ | $+a_1a_3a_6^3$ | $-2a_2a_4a_6^3$ | | | |
| $+3a_0a_3^2a_6^2$ | $-4a_0a_4a_5^2a_6$ | $-3a_1a_4a_5a_6^2$ | $+a_2a_5^2a_6^2$ | | | |
| $-3a_0a_3a_4a_5a_6$ | $+a_0a_5^4$ | $+a_1a_5^3a_6$ | $+a_3^2a_6^3$ | | | |
| $+a_0a_3a_5^3$ | $+2a_1^2a_6^3$ | $-a_2^2a_6^3$ | | | | |
| $+a_1^2a_4a_6^2$ | $-2a_1a_2a_5a_6^2$ | $+a_2a_3a_5a_6^2$ | | | | |
| $-a_1^2a_5^2a_6$ | $-2a_1a_3a_4a_6^2$ | | | | | |
| $-3a_1a_2a_3a_6^2$ | $+a_1a_3a_5^2a_6$ | | | | | |
| $+a_1a_2a_4a_5a_6$ | $+a_2^2a_4a_6^2$ | | | | | |
| $+a_2^3a_6^2$ | | | | | | |